ON A THEOREM IN THE CALCULUS OF DIFFERENCES

By

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It is a curious and may be considered as an important problem in the Calculus of Differences, to assign an expression for the sum of the series

$$\mathrm{X} = u_n(x+n)^n - u_{n-1} \,.\, \frac{n}{1} \,.\, (x+n-1)^n + u_{n-2} \,.\, \frac{n(n-1)}{1\,.\,2} \,.\, (x+n-2)^n - \&\text{c.}; \qquad (1.)$$

which differs from the series for $\Delta^n x^n$ only by its introducing teh coefficients u, determined by the conditions that

$$u_i = +1,\ \ 0,\ \text{or}\ -1,\ \text{according as}\ x+i>0,\ \ =0,\ \text{or}\ <0. \qquad (2.)$$

These conditions may be expressed by the formula

$$u_i = \frac{2}{\pi}\int_0^\infty \frac{dt}{t}\sin(xt+it); \qquad (3.)$$

and if we observe that

$$\frac{d}{dt}\sin(at+b) = a\sin\left(at+b+\frac{\pi}{2}\right),$$
$$\left(\frac{d}{dt}\right)^n \sin(at+b) = a^n\sin\left(at+b+\frac{n\pi}{2}\right),$$

we shall see that the series (1.) may be put under the form

$$\mathrm{X} = \frac{2}{\pi}\int_0^\infty \frac{dt}{t}\left(\frac{d}{dt}\right)^n \Delta^n \sin\left(xt-\frac{n\pi}{2}\right); \qquad (4.)$$

the characteristic Δ of difference being referred to x. But

$$\Delta\sin(2\alpha x+\beta) = 2\sin\alpha\sin\left(2\alpha x+\beta+\alpha+\frac{\pi}{2}\right),$$
$$\Delta^n\sin(2\alpha x+\beta) = (2\sin\alpha)^n\sin\left(2\alpha x+\beta+n\alpha+\frac{n\pi}{2}\right);$$

therefore, changing t, in (4.) to 2α, we find

$$\mathrm{X} = \int_0^\infty \frac{d\alpha}{\alpha}\,\frac{d^n\mathrm{A}}{d\alpha^n}, \qquad (5.)$$

if we make, for abridgment,

$$\mathrm{A} = \frac{2}{\pi} \sin \alpha^n \sin(2x\alpha + n\alpha). \tag{6.}$$

Again, the process of integration by parts gives

$$\int_0^\infty \frac{d\alpha}{\alpha^i} \frac{d^{n-i+1}\mathrm{A}}{d\alpha^{n-i+1}} = i \int_0^\infty \frac{d\alpha}{\alpha^{i+1}} \frac{d^{n-i}\mathrm{A}}{d\alpha^{n-i}},$$

provided that the function

$$\frac{1}{\alpha^i} \frac{d^{n-i}\mathrm{A}}{d\alpha^{n-i}}$$

vanishes both when $\alpha = 0$ and when $\alpha = \infty$, and does not become infinite for any intermediate value of α, conditions which are satisfied here; we have, therefore, finally,

$$\mathrm{X} = 1\,.\,2\,.\,3\,\ldots\,n \int_0^\infty d\alpha\, \frac{\mathrm{A}}{\alpha^{n+1}}. \tag{7.}$$

Hence, if we make

$$\mathrm{P} = \frac{\mathrm{X}}{1\,.\,2\,.\,3\,\ldots\,n}, \quad \text{and} \quad c = 2x + n, \tag{8.}$$

we shall have the expression

$$\mathrm{P} = \frac{2}{\pi} \int_0^\infty d\alpha \left(\frac{\sin \alpha}{\alpha} \right)^n \frac{\sin c\alpha}{\alpha}, \tag{9.}$$

as a transformation of the formula

$$\left.\begin{aligned} \mathrm{P} = \frac{1}{1\,.\,2\,.\,3\,\ldots\,n\,.\,2^n} \Big\{ (n+c)^n - \frac{n}{1}(n+c-2)^n + \frac{n(n-1)}{1\,.\,2}(n+c-4)^n - \&\text{c.} \\ - (n-c)^n + \frac{n}{1}(n-c-2)^n - \frac{n(n-1)}{1\,.\,2}(n-c-4)^n + \&\text{c.} \Big\}; \end{aligned}\right\} \tag{10.}$$

each partial series being continued only as far as the quantities raised to the nth power are positive. Laplace has arrived at an equivalent transformation, but by a much less simple analysis.