ON GEOMETRICAL NETS IN SPACE

By

William Rowan Hamilton

(Proceedings of the Royal Irish Academy, 7 (1862), pp. 532–582.)

Edited by David R. Wilkins

1999

NOTE ON THE TEXT

The paper *On Geometrical Nets in Space* by Sir William Rowan Hamilton was originally published in the seventh volume of the *Proceedings of the Royal Irish Academy.* The following typographical errors in that text have been corrected in this edition:—

in article [13.], equation (47), F_1 has been corrected to F';

in article [21.], equation (100), a missing overline has been placed over the 3rd occurrence of σ on the line.

in article [33.], '(' has been corrected to '[' in the ternary symbol for the plane $\mathrm{AA}'\mathrm{D}_1\mathrm{C}_1\mathrm{B}_2$;

in article [56.], the final point in the list of points on the third typical line $\mathrm{B}'\mathrm{C}'$ was printed as A^{IX}, but this has been corrected to $\mathrm{A}_1^{\mathrm{IX}}$;

in article [58.], the ternary symbol for the point B_0 on the line $[0\,1\,1]$ was printed as $(\overline{1}\,1\,1)$, but has been corrected to $(1\,\overline{1}\,1)$;

in article [74.], the ternary symbol for the line AA' was printed as $[0\,1\,1]$, but this has been corrected to $[0\,1\,\overline{1}]$;

in article [100.], 'remaked' has been corrected to 'remarked';

in article [123.]: the third point on the line of intersection of the planes ABD and $\mathrm{A}_1\mathrm{B}_1\mathrm{D}_1$ was printed as A''; but this has been corrected to C''.

Also, from article [97.] onwards, the roman superscripts on the points A^{IV}, B^{IV}, C^{IV}, etc. were printed in lower case, as A^{iv}, B^{iv}, C^{iv}, etc., but these superscripts have been changed to uppercase, in conformity with the notation established in the earlier articles. Similarly P_0, P_1, and P_2 in articles [1.] and [2.] were originally printed in normal size uppercase roman, but have been changed in this edition to 'small capitals', in conformity with the notation in the remainder of the paper.

David R. Wilkins

Dublin, June 1999

ON GEOMETRICAL NETS IN SPACE.

Sir William Rowan Hamilton.

Read June 24th, 1861.

[*Proceedings of the Royal Irish Academy*, vol. vii (1862), pp. 532–582.]

[1.] When any five points of space, A B C D E, are given, whereof no four are supposed to be complanar, we can connect any two of them by a right line, and the three others by a plane, and determine the point in which these last intersect each other: *deriving* thus a system of *ten lines* Λ_1, *ten planes* Π_1, and *ten points* P_1, from the *given* system of *five points* P_0, by what may be called a *First Construction*.

We may next propose to determine all the new and distinct lines Λ_2, and planes Π_2, which connect the ten derived points P_1, with the five given points P_0, and with each other, and may then inquire what new and distinct points P_2 arise, as intersections* $\Lambda \cdot \Pi$ of lines and planes already obtained: all *such* new lines, planes, and points being said to belong to a *Second Construction*. And then we might proceed, on the same plan, to a *Third* Construction, and to indefinitely many others following: building up thus what Professor *Möbius*, in his *Barycentric Calculus*,† has proposed to call a *Geometric Net in Space*.

[2.] In general, if n denote five or any greater number of *independent* points of space, the number of the derived points of the *form* $\Lambda \cdot \Pi$, or $\mathrm{AB} \cdot \mathrm{CDE}$, which can be obtained by what is relatively to *them* a First Construction, of the kind just now described, is easily seen to be the function,

$$f(n) = \frac{n(n-1)}{2} \cdot \frac{(n-2)(n-3)(n-4)}{2 \,.\, 3};$$

so that $f(5) = 10$, as above, but $f(15) = 30030$. If then the *fifteen points* P_0, P_1 were thus *independent*, or *unconnected* with each other, we might expect to find that the number of points P_2 *derived* from them, at the next stage, should *exceed thirty thousand*. And although it was obvious that many *reductions* of this number must occur, on account of the *dependence* of the ten points P_1 on the five points P_0, yet when I happened to feel a curiosity, some time ago, to determine the precise *number* of those which have been above called *Points of Second*

* Intersections $\Lambda \cdot \Lambda$ of *line with line* (when complanar) are *included* in this class $\Lambda \cdot \Pi$; and intersections $\Pi \cdot \Pi \cdot \Pi$ of *three distinct planes*, when *not* included at this stage, may be reserved for a *subsequent construction*, in which they naturally offer themselves, as of the standard form $\Lambda \cdot \Pi$.

† Der Calcul Barycentrische, Leipzig, 1827, p. 291. Some first results connected with the subject were given, according to the writer's recollection, in a Memoir by *Carnot* on *Transversals*, to which he cannot at present refer.

Construction, and to assign their chief geometrical relations to each other, and to the fifteen former points, it must be confessed that I thought myself about to undertake the solution of a rather formidable Problem. But the motive which led me to attack that problem, namely the desire to try the efficiency of a certain system of *Quinary Symbols*, for points, lines, and planes in space, which the *Method of Vectors* had led me to invent, inspired me with a hope, which I trust that the result of the attempt has not altogether failed to justify. And, in the present communication, I wish first to present some outline of what may be called perhaps a *Quinary Calculus*, before proceeding to give, in the second place, some sketch of the results of its application to the geometrical *Net in Space*.

Part I.—*On a Quinary Calculus for Space.*

[3.] Let ABCDE be (as in [1.]) any five given points of space, whereof no four are situated in any common plane; then, by decomposing ED in the directions of EA, EB, EC, we can always obtain an equation of the form,

$$a\,.\,\mathrm{EA} + b\,.\,\mathrm{EB} + c\,.\,\mathrm{EC} + d\,.\,\mathrm{ED} = 0, \tag{1}$$

in which the coefficients $a\,b\,c\,d$ have determined ratios. And if we next introduce a fifth coefficient e, such that

$$a + b + c + d + e = 0, \tag{2}$$

and add to (1) the identity

$$(a + b + c + d + e)\,\mathrm{OE} = 0, \tag{3}$$

in which O is any arbitrary point (or origin of vectors), we arrive at the following equivalent but more symmetric form,

$$a\,.\,\mathrm{OA} + b\,.\,\mathrm{OB} + c\,.\,\mathrm{OC} + d\,.\,\mathrm{OD} + e\,.\,\mathrm{OE} = 0, \tag{4}$$

in which $a\,b\,c\,d\,e$ may be called the *five (numerical) constants* of the given system of *five points*, A ... E, although only their *ratios* are important, and (as above) their *sum* is *zero*.

[4.] Let P be any other point of space, and let $x\,y\,z\,w\,v$ be coefficients satisfying the equation,

$$(x - v)a\,.\,\mathrm{PA} + (y - v)b\,.\,\mathrm{PB} + (z - v)c\,.\,\mathrm{PC} + (w - v)d\,.\,\mathrm{PD} = 0; \tag{5}$$

then, adding the identity,

$$v(a\,.\,\mathrm{PA} + b\,.\,\mathrm{PB} + c\,.\,\mathrm{PC} + d\,.\,\mathrm{PD} + e\,.\,\mathrm{PE}) = 0, \tag{6}$$

which results from (4), we obtain this other symmetric formula,

$$xa\,.\,\mathrm{PA} + yb\,.\,\mathrm{PB} + zc\,.\,\mathrm{PC} + wd\,.\,\mathrm{PD} + ve\,.\,\mathrm{PE} = 0, \tag{7}$$

which may also be thus written,

$$\mathrm{OP} = \frac{xa\,.\,\mathrm{OA} + yb\,.\,\mathrm{OB} + zc\,.\,\mathrm{OC} + wd\,.\,\mathrm{OD} + ve\,.\,\mathrm{OE}}{xa + yb + zc + wd + ve}, \tag{8}$$

O being again an arbitrary origin; and the *five new and variable coefficients*, $x\,y\,z\,w\,v$, whereof the *ratios of the differences* determine the *position of the point* P, when the five points A ... E are given, may be called the *Quinary Coordinates of that Point* P, with respect to the given system of five points.

[5.] Under these conditions, we may agree to write, briefly,

$$\text{P} = (x, y, z, w, v), \quad \text{or even} \quad \text{P} = (x\,y\,z\,w\,v), \tag{9}$$

whenever it seems that the omission of the commas will not give rise to any confusion; and may call this form a *Quinary Symbol of the Point* P. But because (as above) only the ratios of the differences of the coefficients or coordinates are important, we may establish the following *Formula of Quinary Congruence*, between two *equivalent Symbols* of one *common point*,

$$(x'\,y'\,z'\,w'\,v') \equiv (x\,y\,z\,w\,v), \tag{10}$$

$$\text{if} \quad x' - v' : y' - v' : z' - v' : w' - v' = x - v : y - v : z - v : w - v; \tag{11}$$

reserving the *Quinary Equation*,

$$(x'\,y'\,z'\,w'\,v') = (x\,y\,z\,w\,v), \tag{12}$$

to imply the coexistence of the *five* separate and ordinary equations,

$$x' = x, \quad y' = y, \quad z' = z, \quad w' = w, \quad v' = v. \tag{13}$$

We shall also adopt, as abridgments of notation, the formulæ,

$$t(x, y, z, w, v) = (tx, ty, tz, tw, tv); \tag{14}$$

$$(x' \ldots v') \pm (x \ldots v) = (x' \pm x, \ldots v' \pm v); \tag{15}$$

and shall find it convenient to employ occasionally what may be called the *Quinary Unit Symbol*,

$$\text{U} = (1\,1\,1\,1\,1); \tag{16}$$

although *this* symbol represents *no determined point*, because both the denominator and numerator of the expression (8) vanish, by (2) and (4), when the five coefficients $x\,y\,z\,w\,v$ become each equal to unity.

[6.] With these notations, if Q and Q′ be any *other* quinary symbols, and t and u any two coefficients, we shall have the congruence,

$$\text{Q}' \equiv \text{Q}, \quad \text{if} \quad \text{Q}' = t\text{Q} + u\text{U}; \tag{17}$$

the *two points* P and P′, which are denoted by these *two symbols*, in this case *coinciding*. Again the equation

$$\text{Q}'' = t\text{Q} + t'\text{Q}' + u\text{U}, \tag{18}$$

is found to express that Q, Q′ Q″ are symbols of *three collinear points*; and the *complanarity of four points*, of which the symbols are Q, Q′, Q″, Q‴, is expressed by this other equation of the same form,

$$\text{Q}''' = t\text{Q} + t'\text{Q}' + t''\text{Q}'' + u\text{U}. \tag{19}$$

[7.] If then a *variable point* P be thus *complanar* with *three given points*, P_0, P_1, P_2, its coordinates [4.] must be connected with theirs, by five equations of the form,

$$x = t_0 x_0 + t_1 x_1 + t_2 x_2 + u; \quad \ldots \quad v = t_0 v_0 + t_1 v_1 + t_2 v_2 + u; \tag{20}$$

whence, by elimination of the four arbitrary coefficients $t_0\, t_1\, t_2\, u$, a *linear equation* is obtained, of the form

$$lx + my + nz + rw + sv = 0, \tag{21}$$

with the general relation

$$l + m + n + r + s = 0 \tag{22}$$

between its coefficients; and this equation (21) may be said to be the *Quinary Equation of the Plane* $P_0\, P_1\, P_2$. The five new coefficients $l\, m\, n\, r\, s$ may be called the *Quinary Coordinates of that Plane*; and the plane itself may be denoted by the *Quinary Symbol*,

$$\Pi = [l, m, n, r, s], \quad \text{or briefly,} \quad \Pi = [l\, m\, n\, r\, s], \tag{23}$$

when the commas can be omitted without confusion.

If $R, R', \ldots$ be symbols of this form, for planes $\Pi, \Pi', \ldots$, then the equation

$$R' = tR, \tag{24}$$

in which t is an arbitrary coefficient, expresses that the *two planes* Π, Π' *coincide*; the equation

$$R'' = tR + t'R' \tag{25}$$

expresses that the *three planes* Π, Π', Π'' are *collinear*, or that the *third* passes *through the line of intersection* of the *other two*; and the equation

$$R''' = tR + t'R' + t''R'' \tag{26}$$

expresses that the *four planes* Π, Π', Π'', Π''' are *compunctual* (or *concurrent*), or that the *fourth* passes *through the point of intersection* of the *other three.*

[8.] It is easy to conceive how problems respecting *intersections of lines and planes* can be resolved, on the foregoing principles. And if we define that a point P, or plane Π, is a *Rational Point*, or a *Rational Plane* of the *System* determined by the *five given Points* A ... E, or that it is *rationally related* to those five points, when its *coordinates* are equal (or proportional) to *whole numbers*, it is obvious, from the nature of the *eliminations* employed, that a *plane* which is determined as containing *three rational points*, or a *point* which is determined as the intersection of *three rational planes*, is itself, in the above sense, *rational.* We may also say that a *right line* Λ is a *Rational Line*, when it is the line $P^{\frown}P$ which *connects* two rational points, or the *intersection* $\Pi \cdot \Pi$ of two rational planes: and then the intersection of a rational line with a rational plane, or of two complanar and rational lines with each other, will be a rational point.

[9.] When any two points, P, P$'$, or any two planes Π, Π', have symbols which differ only by the *arrangement* or (*order*) of the five coefficients or coordinates in each, those points, or those planes, may then be said to have one *common type*; or briefly, to be *syntypical.* For example, the five *given* points are thus syntypical, because (omitting commas, as in [5.]) their symbols are,

$$\text{A} = (1\,0\,0\,0\,0), \quad \text{B} = (0\,1\,0\,0\,0), \quad \text{C} = (0\,0\,1\,0\,0), \quad \text{D} = (0\,0\,0\,1\,0), \quad \text{E} = (0\,0\,0\,0\,1). \tag{27}$$

In general, any two syntypical points, or planes, admit of being *derived* from the five given points, by precisely *similar processes of construction*, the *order* only of the *data* being *varied*; and in the *most general case*, a *single type* includes 120 *distinct points*, or *distinct planes*, although this *number* may happen to be diminished, even when the coordinates are all unequal: for example, the type (1 2 3 4 5) includes only *sixty* distinct points, because, by (17), we have in this case the congruence,

$$(1\,2\,3\,4\,5) \equiv (5\,4\,3\,2\,1). \tag{28}$$

[10.] The *anharmonic function* of any group of four collinear points A B C D being denoted by the symbol (A B C D), and defined by the equation,

$$(\text{A B C D}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DA}} = \frac{\text{AB}}{\text{CB}} : \frac{\text{AD}}{\text{CD}}, \tag{29}$$

it will be found that if $\text{P}_0 \ldots \text{P}_3$ be thus *any four collinear points*, of which therefore, by (18), the quinary symbols $Q_0 \ldots Q_3$ are connected by two linear relations, of the forms,

$$Q_1 = t_0 Q_0 + t_2 Q_2 + uU, \quad Q_3 = t'_0 Q_0 + t'_2 Q_2 + u'U, \tag{30}$$

then the *anharmonic of this group of points* is given by the formula,

$$(\text{P}_0\,\text{P}_1\,\text{P}_2\,\text{P}_3) = \frac{t_2 t'_0}{t_0 t'_2}, \tag{31}$$

of which the applications are numerous and important.

And in like manner, if $\Pi_0 \ldots \Pi_3$ be *any four collinear planes*, of which consequently, by (25), the symbols $R_0 \ldots R_3$ are connected by two other linear relations, such as

$$R_1 = t_0 R_0 + t_2 R_2, \quad R_3 = t'_0 R_0 + t'_2 R_2, \tag{32}$$

we have then this other very useful formula of the same kind, of the *anharmonic of this pencil of planes*,

$$(\Pi_0\,\Pi_1\,\Pi_2\,\Pi_3) = \frac{t_2 t'_0}{t_0 t'_2}; \tag{33}$$

it being understood that the anharmonic function of such a *pencil* is the same as that of the *group* of *points*, in which its *planes* are *cut* by any rectilinear *transversal*: so that we may write generally, for *any six points* A ... F, the formula,

$$(\text{E F . A B C D}) = (\text{A}'\,\text{B}'\,\text{C}'\,\text{D}'), \tag{34}$$

if any transversal GH cut the four planes EFA, ... EFD in the four points A$'$, ... D$'$; or in symbols, if

$$\text{A}' = \text{GH} \cdot \text{EFA}, \quad \ldots \quad \text{D}' = \text{GH} \cdot \text{EFD}. \tag{35}$$

[11.] The expression of fractional form,

$$\varphi(x\,y\,z\,w\,v) = \frac{l'x + m'y + n'z + r'w + s'v}{lx + my + nz + rw + sv} = \frac{f'}{f}, \tag{36}$$

in which the ten coefficients $l \ldots s$ and $l' \ldots s'$, are supposed to be given, and to be such (comp. (22)) that

$$l + \cdots + s = 0, \quad \text{and} \quad l' + \cdots + s' = 0, \tag{37}$$

may represent the quotient of any two linear and homogeneous functions, f and f', of the coordinates $x \ldots v$ of a variable point P, or rather of the *differences* of those coordinates (comp. [5.]); and if we assign any *particular* or *constant value*, such as k, to this *quotient*, or *fractional function*, φ, the equation so obtained will represent (comp. (21)) a *plane locus* for that point, which *plane* Π will always pass *through a given line* Λ, determined by equating separately the denominator and numerator of φ to zero. Hence the *four equations*,

$$f = 0, \quad f' = f, \quad f' = 0, \quad f' = kf, \tag{38}$$

which answer to the four values,

$$\varphi = \infty, \quad \varphi = 1, \quad \varphi = 0, \quad \varphi = k, \tag{39}$$

represent a *pencil of four planes* $\Pi_0 \ldots \Pi_3$, of which the quinary symbols (23) may be thus written:—

$$R_0 = [l\,m\,n\,r\,s]; \quad R_2 = [l'\,m'\,n'\,r'\,s']; \quad R_1 = R_2 - R_0; \quad R_3 = R_2 - kR_0; \tag{40}$$

and of which the *anharmonic* is consequently, by (33), the same *quotient*,

$$(\Pi_0\,\Pi_1\,\Pi_2\,\Pi_3) = (k = \varphi =)\frac{f'}{f}, \tag{41}$$

as before. We have therefore this *Theorem*:—

"*The Quotient of any two given homogeneous and linear Functions, of the Differences of the Quinary Coordinates of a variable Point in Space, can always be expressed as the Anharmonic of a Pencil of Planes, whereof three are given, while the fourth passes through the variable Point, and through a given Right Line, which is common to the three former Planes.*"

[12.] For example, we find thus that

$$\frac{x-v}{w-v} = (\text{BC.AEDP}); \quad \frac{y-v}{w-v} = (\text{CA.BEDP}); \quad \frac{z-v}{w-v} = (\text{AB.CEDP}); \tag{42}$$

and that

$$\frac{x-v}{y-v} = (\text{CD.AEBP}); \quad \frac{y-v}{z-v} = (\text{AD.BECP}); \quad \frac{z-v}{x-v} = (\text{BD.CEAP}); \tag{43}$$

the product of these three last anharmonics of pencils being therefore equal to positive unity, so that we have, for *any six points of space*, A B C D E F, the general equation,

$$(\text{AD} \,.\, \text{BECF}) \,.\, (\text{BD} \,.\, \text{CEAF}) \,.\, (\text{CD} \,.\, \text{AEBF}) = 1. \tag{44}$$

If then we *suppress the fifth coefficient, v, in the quinary symbol* (9) *of a point* P, which comes to first substituting, as the congruence (10) permits, the differences $x - v$, $y - v$, $z - v$, $w - v$, and $v - v$ or 0, for x, y, z, w, and v, and then writing simply $x, \ldots w$ instead of $x - v, \ldots w - v$, and omitting the final zero, whereby the quinary symbol (0 0 0 0 1) for the fifth given point E (27) becomes first $(-1, -1, -1, -1, 0)$, or (1 1 1 1 0), and then is reduced to the *quaternary unit symbol* (1 1 1 1), we shall *fall back on that system of anharmonic coordinates in space*, of which some account was given in a former communication* to this Academy: the *anharmonic* (or *quaternary*) *symbol of a plane* Π being, in like manner, *derived from the quinary symbol* (23), by simply *suppressing the fifth coefficient*, or coordinate, s. *Anharmonic coordinates*, whether for *point* or for *plane*, are therefore *included in quinary ones*; but although they have some advantages of *simplicity*, it appears that their *less perfect symmetry*, of reference to the *five given points* A ... E, renders them less adapted to investigations respecting the *Geometrical Net in Space*, which is constructed with those *five* points as data: and that therefore they are less fit than *quinary* coordinates for the purposes of the present paper.

[13.] Retaining then the *quinary form*, we may next observe that although, *when the five coefficients* $l \ldots s$ *are given*, as in [7.], and the *coordinates* $x \ldots v$ of a *point* P are *variable*, the *linear equation* $lx + \cdots + sv = 0$ (21) may be said to be the *Local Equation of a Plane*, namely of the *plane* $[l \ldots s]$, considered as the *locus of the point* $(x \ldots v)$; yet if, on the contrary, we *now* regard $x \ldots v$ as *given*, and $l \ldots s$ as *variable*, the *same linear equation* (21) expresses the *condition* necessary, in order that a *variable plane* $[l \ldots s]$ may pass *through a given point* $(x \ldots v)$; and *in this view*, the formula (21) may be considered to be the *Tangential Equation of that given Point.* Thus the very simple equation,

$$l = 0, \tag{45}$$

expresses the condition requisite for the plane $[l \ldots s]$ passing through the given point (1 0 0 0 0), or A (27); and it is, in that sense, the tangential *equation of that point*: while $m = 0$ is, in like manner, the equation of B, &c. This being understood, if we suppose that F and F′ denote two given, linear, and homogeneous functions of the coordinates $l \ldots s$ of a variable plane Π, we may consider the four equations,

$$\text{F} = 0, \quad \text{F}' = \text{F}, \quad \text{F}' = 0, \quad \text{F}' = k\text{F}, \tag{46}$$

as the tangential equations of *four collinear points*, P_0, P_1, P_2, P_3, whereof the three first are entirely given, but the fourth varies with the value of the coefficient k, although always remaining on the line Λ of the other three; and it is easy to deduce, from the formula (31), by reasonings analogous to those employed in [11.], the following *anharmonic of the group*:

$$(\text{P}_0\,\text{P}_1\,\text{P}_2\,\text{P}_3) = k = \frac{\text{F}'}{\text{F}}. \tag{47}$$

We have therefore this new *Theorem*, analogous to one lately stated:—

* See the Proceedings for the Session of 1859–60.

"*The Quotient of any two given, homogeneous, and linear Functions, of the Quinary Coordinates of a variable Plane, may always be expressed as the Anharmonic of a Group of Points; whereof three are given and collinear, while the fourth is the Intersection of the variable Plane with the given Line on which the other three are situated.*"

[14.] For example, if we wish in this way to *interpret the quotient* $m : n$, of these two coordinates of a *variable plane* Π, or $[l\,m\,n\,r\,s]$ (23), as denoting the *anharmonic of a group of points*, the first three points P_0, P_1, P_2 of that group (47) have here for their tangential equations,

$$n = 0, \quad m - n = 0, \quad m = 0, \tag{48}$$

whereof the *third* has recently been seen [13.] to represent the given point B, and the *first* represents in like manner another given point, namely C, of the initial system: while the *second* represents the point $(0, 1, -1, 0, 0)$, or briefly $(0\,1\,\overline{1}\,0\,0)$, if, to save commas, we write $\overline{1}$ for -1. To *construct* this last point, let us write

$$\mathrm{A}' = (0\,1\,1\,0\,0) \equiv (1\,0\,0\,1\,1), \quad \textit{and} \quad \mathrm{A}'' = (0\,1\,\overline{1}\,0\,0); \tag{49}$$

then, by (18), these two new points A′ and A″ are each *collinear* with B, C, or are on the line BC; and they are, with respect to that line (or to its extreme points) *harmonically conjugate* to each other, because the formula (31) gives easily, by the *first* symbol for A′, the *harmonic equation*,

$$(\mathrm{B\,A'\,C\,A''}) = -1; \tag{50}$$

but also the *second* (or congruent) symbol for A′ shows, by (19), that A′ is in the *plane* ADE; we may therefore write the *formula of intersection*,

$$\mathrm{A}' = \mathrm{BC} \cdot \mathrm{ADE}, \tag{51}$$

whereby the point A′ is entirely determined; and then the point A″, as being its harmonic conjugate with respect to B and C, or as satisfying the equation (50), is to be considered as being itself a known point. We have thus assigned the three first points P_0, P_1, P_2, of the *group* (47), namely the points C, A″, B; and if we denote by L the point $\mathrm{BC} \cdot \Pi$ in which the variable plane Π, or $[l \ldots s]$, intersects the given line BC, so that

$$\mathrm{L} = (0, n, -m, 0, 0), \quad \text{or briefly,} \quad \mathrm{L} = (0\,n\,\overline{m}\,0\,0), \tag{52}$$

writing $\overline{m}$ for $-m$, then the fourth point P_3 is L; and the required *formula of interpretation* for the quotient $m : n$ becomes,

$$\frac{m}{n} = (\mathrm{C\,A''\,B\,L}). \tag{53}$$

In like manner, if we write

$$\mathrm{B}' = (1\,0\,1\,0\,0), \quad \mathrm{C}' = (1\,1\,0\,0\,0), \quad \mathrm{B}'' = (\overline{1}\,0\,1\,0\,0), \quad \mathrm{C}'' = (1\,\overline{1}\,0\,0\,0), \tag{54}$$

and

$$\mathrm{M} = (\overline{n}\,0\,l\,0\,0), \quad \mathrm{N} = (m\,\overline{l}\,0\,0\,0), \tag{55}$$

in which $\overline{n} = -n$ and $\overline{l} = -l$, so that $\text{M} = \text{CA} \cdot \Pi$, $\text{N} = \text{AB} \cdot \Pi$, and

$$\text{B}' = \text{CA} \cdot \text{BDE}, \quad \text{C}' = \text{AB} \cdot \text{CDE}, \quad (\text{C}\,\text{B}'\,\text{A}\,\text{B}'') = (\text{A}\,\text{C}'\,\text{B}\,\text{C}'') = -1, \tag{56}$$

we shall have these two other formulæ of interpretation, analogous to (53),

$$\frac{n}{l} = (\text{A}\,\text{B}''\,\text{C}\,\text{M}), \quad \frac{l}{m} = (\text{B}\,\text{C}''\,\text{A}\,\text{N}); \tag{57}$$

and therefore,

$$(\text{A}\,\text{B}''\,\text{C}\,\text{M}) \cdot (\text{B}\,\text{C}''\,\text{A}\,\text{N}) \cdot (\text{C}\,\text{A}''\,\text{B}\,\text{L}) = 1. \tag{58}$$

[15.] Again, if we denote by Q, R, S the intersections $\text{DA} \cdot \Pi$, $\text{DB} \cdot \Pi$, $\text{DC} \cdot \Pi$, so that

$$\text{Q} = (\overline{r}\,0\,0\,l\,0), \quad \text{R} = (0\,\overline{r}\,0\,m\,0), \quad \text{S} = (0\,0\,\overline{r}\,n\,0), \tag{59}$$

where $\overline{r} = -r$; if also we introduce seven new points syntypical [9.] with the three points $\text{A}'\,\text{B}'\,\text{C}'$, and seven others syntypical with $\text{A}''\,\text{B}''\,\text{C}''$, as follows:

$$\text{A}_1 = (1\,0\,0\,0\,1), \quad \text{B}_1 = (0\,1\,0\,0\,1), \quad \text{C}_1 = (0\,0\,1\,0\,1), \quad \text{D}_1 = (0\,0\,0\,1\,1); \tag{60}$$

$$\text{A}_2 = (1\,0\,0\,1\,0), \quad \text{B}_2 = (0\,1\,0\,1\,0), \quad \text{C}_2 = (0\,0\,1\,1\,0); \tag{61}$$

$$\text{A}'_1 = (1\,0\,0\,0\,\overline{1}), \quad \text{B}'_1 = (0\,1\,0\,0\,\overline{1}), \quad \text{C}'_1 = (0\,0\,1\,0\,\overline{1}), \quad \text{D}'_1 = (0\,0\,0\,1\,\overline{1}); \tag{62}$$

$$\text{A}'_2 = (1\,0\,0\,\overline{1}\,0), \quad \text{B}'_2 = (0\,1\,0\,\overline{1}\,0), \quad \text{C}'_2 = (0\,0\,1\,\overline{1}\,0); \tag{63}$$

so that, by principles already established, we shall have the seven relations of intersection,

$$\text{A}_1 = \text{EA} \cdot \text{BCD}, \quad \text{B}_1 = \text{EB} \cdot \text{CAD}, \quad \text{C}_1 = \text{EC} \cdot \text{ABD}, \quad \text{D}_1 = \text{ED} \cdot \text{ABC}, \tag{64}$$

$$\text{A}_2 = \text{DA} \cdot \text{BCE}, \quad \text{B}_2 = \text{DB} \cdot \text{CAE}, \quad \text{C}_2 = \text{DC} \cdot \text{ABE}, \tag{65}$$

and the seven harmonic relations,

$$(\text{E}\,\text{A}_1\,\text{A}\,\text{A}'_1) = (\text{E}\,\text{B}_1\,\text{B}\,\text{B}'_1) = (\text{E}\,\text{C}_1\,\text{C}\,\text{C}'_1) = (\text{E}\,\text{D}_1\,\text{D}\,\text{D}'_1) = -1, \tag{66}$$

$$(\text{D}\,\text{A}_2\,\text{A}\,\text{A}'_2) = (\text{D}\,\text{B}_2\,\text{B}\,\text{B}'_2) = (\text{D}\,\text{C}_2\,\text{C}\,\text{C}'_2) = -1, \tag{67}$$

by means of which 14 last relations these 14 new points can all be geometrically constructed; we shall then be able to interpret, on the recent plan [13.], the three new quotients, $l : r$, $m : r$, $n : r$, as anharmonics of groups, as follows:

$$\frac{l}{r} = (\text{D}\,\text{A}'_2\,\text{A}\,\text{Q}); \quad \frac{m}{r} = (\text{D}\,\text{B}'_2\,\text{B}\,\text{R}); \quad \frac{n}{r} = (\text{D}\,\text{C}'_2\,\text{C}\,\text{S}); \tag{68}$$

with the analogous interpretations,

$$\frac{l}{s} = (\text{E}\,\text{A}'_1\,\text{A}\,\text{X}); \quad \frac{m}{s} = (\text{E}\,\text{B}'_1\,\text{B}\,\text{Y}); \quad \frac{n}{s} = (\text{E}\,\text{C}'_1\,\text{C}\,\text{Z}); \quad \frac{r}{s} = (\text{E}\,\text{D}'_1\,\text{D}\,\text{W}), \tag{69}$$

if X, Y, Z, W denote the intersections $\text{EA} \cdot \Pi$, $\text{EB} \cdot \Pi$, $\text{EC} \cdot \Pi$, $\text{ED} \cdot \Pi$, so that

$$\text{X} = (\overline{s}\,0\,0\,0\,l), \quad \text{Y} = (0\,\overline{s}\,0\,0\,m), \quad \text{Z} = (0\,0\,\overline{s}\,0\,n), \quad \text{W} = (0\,0\,0\,\overline{s}\,r), \tag{70}$$

where $\overline{s} = -s$.

[16.] As regards the *notations* employed, it may be observed that although we have often, as in (9) or (27), &c., *equated a point*, or rather its *literal symbol*, A or P, &c., to the *corresponding quinary symbol* (1 0 0 0 0) or $(x\,y\,z\,w\,v)$, &c., of that point, yet in some formulæ, such as (17) (18) (19), in which we had occasion to treat of *linear combinations* of such quinary symbols, we substituted *new letters*, such as Q, Q', for P, P′, &c., in order to avoid the apparent strangeness of writing such expressions* as $t\mathrm{P} + t'\mathrm{P}'$, &c. To *economise symbols*, however, we may agree to *retain the literal symbols first used*, for any system of given or derived points, but to *enclose them in parentheses*, when we wish to employ them as *denoting quinary symbols in combination with each other*; writing, at the same time, for the sake of uniformity (U) instead of U, as the *quinary unit symbol* (16). And thus, if we agree also that an *equation* between *two unenclosed* and *literal symbols of points*, P and P′, shall be understood as expressing that the two points so denoted *coincide*, we may write anew those formulae (17) (18) (19) as follows:

$$\mathrm{P}' = \mathrm{P}, \text{ if } (\mathrm{P}') = t(\mathrm{P}) + u(\mathrm{U}); \tag{71}$$

$$\mathrm{P}'' \textit{ on line } \mathrm{PP}', \text{ if } (\mathrm{P}'') = t(\mathrm{P}) + t'(\mathrm{P}') + u(\mathrm{U}); \tag{72}$$

$$\mathrm{P}''' \textit{ in plane } \mathrm{PP}'\mathrm{P}'', \text{ if } (\mathrm{P}''') = t(\mathrm{P}) + t'(\mathrm{P}') + t''(\mathrm{P}'') + u(\mathrm{U}). \tag{73}$$

[17.] We may also occasionally denote a point *in the given plane* of A, B, C by the *ternary symbol*,

$$(x, y, z), \quad \text{or} \quad (x\,y\,z), \tag{74}$$

considered here as an *abridgment* of the *quinary* symbol $(x\,y\,z\,0\,0)$; and the *right line* which is the *trace on that plane*, of any *other plane* Π, or $[l\,m\,n\,r\,s]$ (23) may be denoted by this *other ternary symbol*,

$$[l, m, n], \quad \text{or} \quad [l\,m\,n]; \tag{75}$$

these two last ternary symbols being *connected* by the relation,

$$lx + my + nz = 0, \tag{76}$$

if the *point* $(x\,y\,z)$ be *on the line* $[l\,m\,n]$. And the *point* P in which any *other* line Λ, *not* situated in the plane ABC, *intersects* that *plane*, may be said to be the *trace* of that *line*.

[18.] For example, the *point* D_1 is, by (64), the *trace of the line* DE; and if we write,

$$\mathrm{A}_0 = (\bar{1}\,1\,1), \quad \mathrm{B}_0 = (1\,\bar{1}\,1), \quad \mathrm{C}_0 = (1\,1\,\bar{1}), \tag{77}$$

then these three points are the respective traces of the three lines $\mathrm{A}_1\mathrm{A}_2$, $\mathrm{B}_1\mathrm{B}_2$, $\mathrm{C}_1\mathrm{C}_2$; because they are, by the notation (74), in the given plane, and we have, by (60) and (61), the three following symbolical equations of the form (72),

$$(\mathrm{A}_0) + (\mathrm{A}_1) + (\mathrm{A}_2) = (\mathrm{B}_0) + (\mathrm{B}_1) + (\mathrm{B}_2) = (\mathrm{C}_0) + (\mathrm{C}_1) + (\mathrm{C}_2) = (\mathrm{U}), \tag{78}$$

* Expressions of this *form* occur continually in the *Barycentric Calculus* of *Moebius*, but with significations entirely different from those here proposed.

which expresses the three collineations, $A_0A_1A_2$, $B_0B_1B_2$, $C_0C_1C_2$.

We have also the three other collineations, AD_1A', BD_1B', CD_1C', because the quinary symbols (27) (49) (54) (60) give the equations,

$$(A) + (A') + (D_1) = (B) + (B') + (D_1) = (C) + (C') + (D_1) = (U); \tag{79}$$

and these *three lines*, $AA'D_1$, &c., are the *traces of the three planes* ADE, BDE, CDE, of which *planes* the respective *equations* (21), and *quinary symbols* (23), are

$$y - z = 0, \quad z - x = 0, \quad x - y = 0, \tag{80}$$

and

$$[0\,1\,\bar{1}\,0\,0], \quad [\bar{1}\,0\,1\,0\,0], \quad [1\,\bar{1}\,0\,0\,0]; \tag{81}$$

so that the *ternary symbols* of the three last *lines*, regarded as their *traces*, are simply, by (75),

$$[0\,1\,\bar{1}], \quad [\bar{1}\,0\,1], \quad [1\,\bar{1}\,0]. \tag{82}$$

Accordingly, whether we consider the point $A = (1\,0\,0)$, or $A' = (0\,1\,1)$, or $D_1 = (1\,1\,1)$, (this *ternary* symbol of D_1 being *congruent* to the former *quinary* symbol $(0\,0\,0\,1\,1)$ for that point (60),) we have in each case the relation $y - z = 0$ between its coordinates; and similarly for the two other lines.

[19.] As other examples, the *four planes*,

$$A_1B_1C_1, \quad A_2B_2C_2, \quad A'_1B'_1C'_1, \quad A'_2B'_2C'_2, \tag{83}$$

have for their quinary equations,

$$x + y + z = 2w + v, \quad x + y + z = w + 2v, \quad x + y + z + v = 4w, \quad x + y + z + w = 4v, \tag{84}$$

and for their quinary symbols,

$$[1\,1\,1\,\bar{2}\,\bar{1}], \quad [1\,1\,1\,\bar{1}\,\bar{2}], \quad [1\,1\,1\,\bar{4}\,1], \quad [1\,1\,1\,1\,\bar{4}]; \tag{85}$$

they have therefore a *common trace*, namely the line

$$[1\,1\,1], \text{ or } A''B''C'', \tag{86}$$

because, by (49) and (54), we may now write

$$A'' = (0\,1\,\bar{1}), \quad B'' = (\bar{1}\,0\,1), \quad C'' = (1\,\bar{1}\,0), \tag{87}$$

and the coordinates of each of these three last points satisfy the equation,

$$x + y + z = 0. \tag{88}$$

Accordingly we have, by (60) (61) (62) (63), the three following sets of symbolical equations of the form (72),

$$\left.\begin{aligned} (A'') &= (B_1) - (C_1) = (B_2) - (C_2) = (B'_1) - (C'_1) = (B'_2) - (C'_2), \\ (B'') &= (C_1) - (A_1) = (C_2) - (A_2) = (C'_1) - (A'_1) = (C'_2) - (A'_2), \\ (C'') &= (A_1) - (B_1) = (A_2) - (B_2) = (A'_1) - (B'_1) = (A'_2) - (B'_2), \end{aligned}\right\} \tag{89}$$

we see that the *point* A'' is the *common trace* of the *four lines*, B_1C_1, B_2C_2, $B'_1C'_1$, $B'_2C'_2$; B'' of C_1A_1, C_2A_2, $C'_1A'_1$, $C'_2A'_2$; and C'' of A_1B_1, A_2B_2, $A'_1B'_1$, $A'_2B'_2$.

[20.] In all such cases as these, in which we have to consider a *set of three points* P, or a *set of three planes* Π, of which the *first* is *geometrically derived* from A B C D E according to the *same rule of construction*, as that according to which the *second* is derived from B C A D E, and the *third* from C A B D E, we can *symbolically derive the second from the first*, and in like manner the *third* from the *second*, (or again the first from the third,) by writing, in each case, the *third*, *first*, and *second coefficients*, or coordinates, in the places of the *first*, *second*, and *third*, respectively. In symbols, we may express the *law of successive derivation*, of certain *syntypical* points or planes [9.] from one another, by the formulæ,

$$\text{if } \mathrm{P}(\mathrm{A\,B\,C}) = (x\,y\,z\,w\,v), \text{ then } \mathrm{P}(\mathrm{B\,C\,A}) = (z\,x\,y\,w\,v), \text{ and } \mathrm{P}(\mathrm{C\,A\,B}) = (y\,z\,x\,w\,v); \tag{90}$$

and if

$$\Pi(\mathrm{A\,B\,C}) = [l\,m\,n\,r\,s], \text{ then } \Pi(\mathrm{B\,C\,A}) = [n\,l\,m\,r\,s], \text{ and } \Pi(\mathrm{C\,A\,B}) = [m\,n\,l\,r\,s]; \tag{91}$$

as has been already exemplified in the systems (27), (60), (61), (62), (63), (77), (81), (87), for *points* or *planes*, and in (82) for *lines*, considered as *traces* of planes. In all these cases, therefore, we can, with perfect clearness and *definiteness* of signification, *abridge the notation*, by writing *only the first* (or indeed *any one*) of the *three* equations (90) or (91) and then appending an "&c."; for the *law* which has been just stated will always enable us to *recover* (or deduce) *the other two*. We may therefore briefly but sufficiently express several of the foregoing results, by writing,

$$\left.\begin{array}{r} \mathrm{A} = (1\,0\,0), \text{ \&c.}; \quad \mathrm{A}' = (0\,1\,1), \text{ \&c.}; \quad \mathrm{A}'' = (0\,1\,\bar{1}), \text{ \&c.}; \quad \mathrm{A}_0 = (\bar{1}\,1\,1), \text{ \&c.}; \\ \mathrm{A}_1 = (1\,0\,0\,0\,1), \text{ \&c.}; \quad \mathrm{A}_2 = (1\,0\,0\,1\,0), \text{ \&c.}; \\ \mathrm{A}_1' = (1\,0\,0\,0\,\bar{1}), \text{ \&c.}; \quad \mathrm{A}_2' = (1\,0\,0\,\bar{1}\,0), \text{ \&c.}; \end{array}\right\} \tag{92}$$

$$\textit{Plane } \mathrm{ADE} = [0\,1\,\bar{1}\,0\,0], \text{ \&c.}; \quad \textit{Line } \mathrm{AD_1A}' = [0\,1\,\bar{1}], \text{ \&c.}; \tag{93}$$

to which we may add these other symbols of planes and lines, each supposed to be followed by an "&c.":

$$\text{plane } \mathrm{BCD} = [1\,0\,0\,0\,\bar{1}]; \quad \mathrm{BCE} = [1\,0\,0\,\bar{1}\,0]; \quad \text{trace} = \mathrm{BC} = [1\,0\,0]; \tag{94}$$

$$\text{plane } \mathrm{DB'B_1C'C_1} = [\bar{1}\,1\,1\,0\,\bar{1}]; \quad \mathrm{EB'B_2C'C_2} = [\bar{1}\,1\,1\,\bar{1}\,0]; \quad \text{trace} = \mathrm{B'C'A''} = [\bar{1}\,1\,1]; \tag{95}$$

$$\text{plane } \mathrm{AB_1C_2C_1B_2} = [0\,1\,1\,\bar{1}\,\bar{1}]; \quad \text{trace} = \mathrm{AA''} = [0\,1\,1]; \tag{96}$$

this line $\mathrm{AA''}$ passing also, by (77), through the two points $\mathrm{B_0}$ and $\mathrm{C_0}$;

$$\text{plane } \mathrm{B_1C_1D_1} = [\bar{2}\,1\,1\,1\,\bar{1}]; \quad \mathrm{B_2C_2D_1} = [\bar{2}\,1\,1\,\bar{1}\,1]; \quad \text{trace} = \mathrm{D_1A''} = [\bar{2}\,1\,1]; \tag{97}$$

$$\left.\begin{array}{l} \text{plane } \mathrm{A'B_1B_2} = [\bar{2}\,\bar{1}\,1\,1\,1]; \quad \text{trace} = \mathrm{A'B_0} = [\bar{2}\,\bar{1}\,1]; \\ \text{plane } \mathrm{A'C_1C_2} = [\bar{2}\,1\,\bar{1}\,1\,1]; \quad \text{trace} = \mathrm{A'C_0} = [\bar{2}\,1\,\bar{1}]; \end{array}\right\} \tag{98}$$

where it may be noticed that the symbol for $\mathrm{A'C_1C_2}$, or for $\mathrm{A'C_0}$, may be deduced from that for $\mathrm{A'B_1B_2}$, or for $\mathrm{A'B_0}$, by simply interchanging the second and third coefficients, or coordinates. It is easy to see that the quinary symbol for the plane ABC itself is on the same plan $[0\,0\,0\,1\,\bar{1}]$, the equation of that plane being $w = v$; and it will be remembered that, by [18.], the ternary symbol for the point $\mathrm{D_1}$ in that plane is $(1\,1\,1)$.

[21.] A *right Line* Λ *in Space* may be regarded in two principal views, as follows. Ist, it may be considered as the *locus of a variable point* P, *collinear with two given points* P_0, P_1; and in this view, the *symbol*

$$t_0(P_0) + t_1(P_1), \qquad \text{(comp. (72),)}$$

for the variable *point* upon the line, may be regarded as a *Local Symbol* (or *Point-Symbol*) *of the Line* Λ *itself.* Thus

$$(0\,t\,t'), \text{ or } (0\,y\,z), \tag{99}$$

may either represent *an arbitrary point on the line* BC; or, *as a local symbol*, that *line itself.* Or IInd, we may consider a line Λ as a *hinge, round which a plane* Π *turns*, so as to be always *collinear* [7.] *with two given planes* Π_0, Π_1 through the line; and then a symbol of the form

$$t_0[\Pi_0] + t_1[\Pi_1], \qquad \text{(comp. (25),)}$$

which represents immediately the *variable plane* Π, may be regarded as being *also a Tangential Symbol* (or *Plane-Symbol*) *for the line* Λ. For example the line BC may be thus represented, not only by the *local* symbol (99), but also by the *tangential* symbol,

$$[\overline{\sigma}\,0\,0\,t\,u], \text{ if } \sigma = t + u, \text{ and } \overline{\sigma} = -\sigma. \tag{100}$$

In fact, this last symbol can be derived, by linear combinations, from the symbols (94) for the two planes BCD, BCE, which intersect in the line BC; and if any particular value be assigned to the ratio $t : u$, a particular *plane through that line* results. But it is time to apply these general principles to the *Geometrical Net in Space.*

Part II.—*Applications to the Net in Space: Enumeration and Classification of the Lines, Planes, and Points of that Net, to the end of the Second Construction.*

[22.] The *data* of the *Geometrical Net* are, by [1.] the *five points* A B C D E, or P_0; of which the *quinary symbols* (27) have been assigned, and shown to be *syntypical* [9.]; and also the *ternary* symbols (92) of the three first of them. Of these the symbol

$$A = (1\,0\,0)$$

may be taken as the *type*; and the point A itself my be said to be a *First Typical Point.*

[23.] The *derived lines* Λ_1 of *First Construction* [1.], are the *ten* following,

BC, &c.; DA, &c.; EA, &c.; and DE;

the "&c." being interpreted as in [20.]; and each line Λ_1 connecting, by this construction, *two* points P_0. Among these the line BC may be selected as a *First Typical Line*; and its *symbols* [21.], namely,

$$(0\,y\,z), \text{ and } [\overline{\sigma}\,0\,0\,t\,u],$$

whereof the former represents this line BC considered as the *locus* of a *variable point*, while the latter represents the same line considered as the *hinge* of a *variable plane*, may be taken as *types* (the *point-type* and the *plane-type*) of the *group* of the *ten lines* Λ_1.

[24.] The *derived planes* Π_1 of *first* construction are in like manner *ten*; namely

ADE, &c.;   BCE, &c.;   BCD, &c.;   and   ABC,

each obtained by connecting *three* points P_0. Of these the last has, by [20.] the quinary *symbol*,

$$\text{ABC} = [0\,0\,0\,1\,\overline{1}],$$

which may be taken as a *type* of the *group* Π_1; and the plane ABC itself may be called a *First Typical Plane.* As a verification, we see that when we make $\sigma = t + u = 0$, in the second symbol [23.], and divide by t, we are led to the recent symbol for ABC, as one of the planes which pass through the line BC.

[25.] The *derived points* P_1, of the same *first construction*, which are all, by [1.], of the form $\Lambda_1 \cdot \Pi_1$, are in like manner *ten*; namely the intersections,

BC · ADE, &c.;   DA · BCE, &c.;   EA · BCD, &c.;   and   DE · ABC,

which have been denoted in [14.] and [15.] by the letters, or *literal symbols*,

A', &c.; A_2, &c.; A_1, &c.;   and D_1,

and for which *quinary symbols* (49) (54) (60) (61) have been assigned. Of these ten points *four*, namely A', B', C', D_1, are situated *in the plane* ABC, and have accordingly been represented [20.] by *ternary symbols* also: and we may take the particular symbol of this sort,

$$A' = (0\,1\,1),$$

as a *type* of this *group* P_1; understanding, however, that the *full* or *quinary type* is to be recovered from this *ternary type*, by *restoring the two omitted zeros*; so that we have, more fully,

$$A' = (0\,1\,1\,0\,0) \equiv (1\,0\,0\,1\,1).$$

And the *point* A' itself may be considered as a *Second Typical Point.*

[26.] We have thus denoted, by *literal* and by *quinary symbols*, whereof some have been *abridged* to *ternary* ones [17.], and have been also represented by *types* [9.], not only the *five given points* P_0, but all the *ten lines* Λ_1, *ten planes* Π_1, and *ten points* P_1, of what has been called, in [1.], the *First Construction.* And it is evident that we have, at this stage, *ten triangles* T_1, namely the ten,

ADE, &c.;   BCE, &c.;   BCD, &c.;   and   ABC,

whereof each is contained in a plane Π_1; and also *five pyramids* R_1, each bounded by *four* of these *triangles*, namely, the pyramids,

BCDE,   CADE,   ABDE,   ABCD,

which may be called the pyramids A, B, C, D, E; each marked by the literal symbol of *that one* of the five points P_0, which is *not a corner* of the pyramid.

[27.] It may be remarked, that *ten arbitrary lines* in space *intersect*, generally, *ten arbitrary planes*, in *one hundred points*; but that this *number* of intersections $\Lambda_1 \cdot \Pi_1$ is *here* reduced to *fifteen*, whereof only *ten* are *new*; because *each* of the *five points* P_0 *counts* as *twelve*, since in each of those points *four lines cut* (each) *three planes*, while *each* of the *ten planes contains three lines*; so that *thirty binary combinations* are *not cases of intersection*, and *sixty* such cases conduct only to the five *old* (or given) points. This sort of *arithmetical verification* of the accuracy of an *enumeration* of *derived points*, or lines, or planes, will be found useful in more complex cases, although it was not necessary here.

[28.] Proceeding to a *Second Construction* [1.], we may begin by determining the *lines* Λ_2, whereof each connects some *two* (at least) of the *fifteen points* P_0, P_1, but *not* any two of the *five* points P_0, since otherwise it would be a line Λ_1. If the 15 points to be connected were *independent*, they would give, generally, by their binary combinations, 105 lines; but the *ten collineations of construction*,

$$\text{BCA}', \text{ \&c.}; \quad \text{DAA}_2, \text{ \&c.}; \quad \text{EAA}_1, \text{ \&c.}; \quad \text{and} \quad \text{EDD}_1,$$

show that 30 of these *combinations* are to be rejected, as giving only the ten old lines. The remaining number, 75, is still farther reduced by the consideration that we have (comp. (79)) the *fifteen derived collineations*,

$$\text{AA}'\text{D}_1, \text{ \&c.}; \quad \text{AB}_1\text{C}_2, \text{ \&c.}; \quad \text{AC}_1\text{B}_2, \text{ \&c.}; \quad \text{DA}'\text{A}_1, \text{ \&c.}; \quad \text{EA}'\text{A}_2, \text{ \&c.};$$

which represent only *fifteen new lines*, of a *group* which we shall denote by $\Lambda_{2,1}$, but *count* (comp. [27.]) as 45 binary combinations of the 15 points. There remain only 30 such combinations to be considered; and these give in fact a *second group*, $\Lambda_{2,2}$, consisting of *thirty lines of second construction*: namely the *thirty edges* of the *five new pyramids* R_2,

$$\text{C}'\text{B}'\text{A}_2\text{A}_1, \quad \text{A}'\text{C}'\text{B}_2\text{B}_1, \quad \text{B}'\text{A}'\text{C}_2\text{C}_1, \quad \text{A}_2\text{B}_2\text{C}_2\text{D}_1, \quad \text{A}_1\text{B}_1\text{C}_1\text{D}_1,$$

which are respectively *inscribed* in the five former pyramids R_1 [26.], and are *homologous* to them, the five given points A … E being the respective *centres of homology*; for example, $\text{C}' = \text{AB} \cdot \text{CDE}$, &c. The corresponding *planes of homology* will present themselves somewhat later, in connexion with the points P_2.

[29.] On the whole, then, there are only *forty-five distinct lines of second construction* Λ_2; and these naturally divide themselves into *two groups*, of 15 lines $\Lambda_{2,1}$, and 30 lines $\Lambda_{2,2}$, as above. *Each* line of the *first group* $\Lambda_{2,1}$ connects *one* point P_0 with *two* points P_1; as each line Λ_1 had connected *one* point P_1 with *two* points P_0, but *no* line of the *second group* $\Lambda_{2,2}$ connects, at this stage of the construction, more than *two* points, which are *both* points P_1. Through *no* point P_0, therefore, can we draw *any line* $\Lambda_{2,2}$; but through *each* point P_0 we can draw *three lines* $\Lambda_{2,1}$; and each of these is determined as the *intersection of two planes* Π_1 through that point, or as *crossing two opposite edges* of that *pyramid* R_1, which has *not* the point P_0 for a corner (comp. [26.]): for example, $\text{AA}'\text{D}_1$ is the intersection of ABC, ADE, and crosses the lines BC, DE. And besides being, as in [28.], the *edges* of certain *other* and *inscribed* pyramids R_2, the 30 lines $\Lambda_{2,2}$ are also the *sides* of *ten new triangles* T_2, namely,

$$\text{D}_1\text{A}_1\text{A}_2,\ \&\text{c.};\quad \text{C}_1\text{B}_1\text{A}',\ \&\text{c.};\quad \text{C}_2\text{B}_2\text{A}',\ \&\text{c.};\quad \text{and}\quad \text{A}'\text{B}'\text{C}',$$

situated *in the ten planes* Π_1, and *inscribed* in the *ten old triangles* T_1, to which also they are *homologous*; the corresponding *centres of homology* being the ten points P_1, in the same order,

$$\text{A}',\ \&\text{c.};\quad \text{A}_2,\ \&\text{c.};\quad \text{A}_1,\ \&\text{c.};\quad \text{and}\quad \text{D}_1,\ \text{as before.}$$

The *axes of homology* of these *ten pairs of triangles* T_1, T_2, will offer themselves a little later, in connexion with points P_2.

[30.] All this may be considered as evident from *geometry* alone, at least with the assistance of *literal symbols*, such as those used above. But to deduce the same things by *calculation*, with *quinary symbols* and *types*, on the plan of the present Paper, we may observe that the symbolical equation,

$$(1\,0\,0\,0\,0) + (0\,1\,1\,0\,0) + (0\,0\,0\,1\,1) = (1\,1\,1\,1\,1),$$

considered as a type of all equations of the same form, proves by (18) or (72) that each point P_1 can, in three different ways, be combined with another point P_1, so that their joining line shall pass through a point P_0; and that thus the *group* of the 15 lines $\Lambda_{2,1}$ arises, of which the line $\text{AA}'\text{D}_1$ is a specimen, and may be called a *Second Typical Line* (the *first* such line having been BC, by [23.]). The *complete* quinary symbol of a *point* on this line is $(t\,u\,u\,v\,v)$, which is however congruent to one of the form $(t\,u\,u\,0\,0)$, and may therefore be abridged to the ternary symbol $(t\,u\,u)$, or $(x\,y\,y)$; and the quinary symbol of a *plane* through the same line is of the form $[0\,m\,\overline{m}\,r\overline{r}]$, or $[0\,t\,\overline{t}\,u\overline{u}]$; we may therefore, by [21.] (comp. [23.]) consider the two expressions,

$$(x\,y\,y),\ \text{and}\ [0\,t\,\overline{t}\,u\overline{u}],$$

as being not only *local and tangential symbols* for the *particular* (or *typical*) *line* $\text{AA}'\text{D}_1$ *itself*, but also *local and tangential types* for the *group* $\Lambda_{2,1}$; or as the *point-type*, and the *plane-type*, of that group.

[31.] The two points P_1, of which the quinary symbols have been thus combined in [30.], had *no common coordinate different from zero*; but there remains to be considered the case, in which two points of that group *have* such a coordinate: for example, when the points have for their symbols,

$$(1\,0\,1\,0\,0)\ \text{and}\ (1\,1\,0\,0\,0),\ \text{or}\ (1\,0\,1)\ \text{and}\ (1\,1\,0).$$

The *point-symbol* and *plane-symbol* of the *line* Λ_2 connecting *these* two points P_1 are easily seen to be (with the same significations of σ and $\overline{\sigma}$ as before),

$$(\sigma\,t\,u\,0\,0),\ \text{or}\ (\sigma\,t\,u),\ \text{and}\ [\overline{t}\,t\,t\,u\overline{\sigma}];$$

but no choice of the arbitrary *ratio*, $t : u$, with $\sigma = t + u$, will reduce the symbol $(\sigma\,t\,u)$ to denote *any one* of the 15 points P_0, P_1, except the *two* points P_1 (in this case B' and C'), by joining which the line is obtained; considering therefore the last two *symbols* as *types*, we see that they represent a *second group*, consisting of *thirty lines* $\Lambda_{2,2}$; but that there can be *no third group*, of *lines* Λ_2 of *second construction.* The *particular line* $B'C'$, which the symbols in the present paragraph represent may be taken as *typical* of this *second group*; and may be called (comp. [23.] and [30.]) a *Third Typical Line* of the System, or *Net*, determined by the five given points A ... E. And the *pyramids* R_1, R_2, and *triangles* T_1, T_2, of first and second constructions, of which the *literal symbols* have been assigned in [26.] [28.] [29.], might also have easily been suggested and studied, by *quinary* symbols and types alone.

[32.] As regards the *Planes* Π_2 of *Second Construction* [1.], it is easily seen that no such plane contains any *two* points P_0, or any *one* line Λ_1; for example, the *first typical line* BC [23.] *contains* the point A'; and if we *connect* it with any one of the four points A, B', C', D_1, we only get a plane Π_1, namely ABC; if with D, A_1, B_2, or C_2, we get another plane Π_1, namely BCD; and if with any one of the four remaining points E, A_2, B_1, C_1, the plane BCE is obtained. Accordingly the general symbol $[\overline{\sigma}\,0\,0\,t\,u]$, in [23.], for a plane through the line BC, gives $\sigma = 0$, or $t = 0$, or $u = 0$, when we seek to particularize it, by the first, the second, or the third of these three sets of conditions respectively.

[33.] But if we take the symbol $[0\,t\,\overline{t}\,u\,\overline{u}]$, in [30.], for a plane through the *second typical line* $AA'D_1$, and seek to particularize *this* symbol by the condition of passing through some one of the eight points P_1 which are not situated upon it, we are conducted to the following results. The points B', C' give $t = 0$, and the points A_1, A_2 give $u = 0$; these points therefore give only two planes Π_1, namely the two planes ABC and ADE, of which the line Λ_2 is the intersection. But the points B_1, C_2 give $t = u$, and the points C_1, B_2 give $t = -u$; *these points* therefore give *two planes* of a *new group* $\Pi_{2,1}$, namely (comp. [20.]) the two following:

$$\text{plane } AA'D_1B_1C_2 = [0\,1\,\overline{1}\,1\,\overline{1}]; \quad \text{plane } AA'D_1C_1B_2 = [0\,1\,\overline{1}\,\overline{1}\,1];$$

which are of the same *type* as the plane (96), namely,

$$\text{plane } AB_1C_2C_1B_2 = [0\,1\,1\,\overline{1}\,\overline{1}].$$

There are *fifteen* such *planes* $\Pi_{2,1}$, as the type sufficiently shows; each passes through *one point* P_0, and contains *two lines* $\Lambda_{2,1}$, containing also *four lines* $\Lambda_{2,2}$; as, for instance, the last-mentioned plane $AB_1C_2C_1B_2$, which we shall call (comp. [24.]) the *Second Typical Plane*, contains the *two* lines AB_1C_2, AC_1B_2 [28.], and the *four* lines B_1C_1, C_1C_2, C_2B_2, B_2B_1; that is to say, the *two diagonals* and the *four sides* of the *quadrilateral* $B_1\,C_1\,C_2\,B_2$, of which the *plane* $\Pi_{2,1}$ passes through A.

[34.] We have now exhausted all the planes Π_2 which contain any point P_0; but there exists a *second group of planes*, $\Pi_{2,2}$, each of which is determined as connecting *three* points P_1, although passing through *no* point P_0. Thus if we take the *third typical line* $B'C'$ [31.], and the symbol $[\overline{t}\,t\,t\,u\,\overline{\sigma}]$ for a plane through it, we get indeed $t = 0$, or a plane Π_1, namely, ABC, if we oblige the plane through $B'C'$ to contain A, or B, or C, or A', or D_1; and we get $u = 0$, or $[\overline{1}\,1\,1\,0\,\overline{1}]$, or a plane $\Pi_{2,1}$, namely $DB'B_1C'C_1$, as in (95), if we oblige it to contain

D, or B_1, or C_1; while we get $\sigma = 0$, or $[\overline{1}\,1\,1\,\overline{1}\,0]$, or $EB'B_2C'C_2$, again as in (95), if we oblige it to contain E, or B_2, or C_2. But there remain the two points A_1 and A_2, determining the two new planes $B'C'A_1$ and $B'C'A_2$, for the former of which we have $t + \sigma = 0$, or $u = -2t$, $\sigma = -t$, and therefore have the symbol $[\overline{1}\,1\,1\,\overline{2}\,1]$; while for the latter we have $u = t$, $\sigma = 2t$, and therefore the syntypical symbol $[\overline{1}\,1\,1\,1\,\overline{2}]$. There are *twenty planes* of this *group* $\Pi_{2,2}$, as may be at once concluded from inspection of the *type*; among which (comp. [19.]) we shall select the following,

$$\text{plane } A_1B_1C_1 = [1\,1\,1\,\overline{2}\,\overline{1}],$$

and call this a *Third Typical Plane.* And it is evident that these 20 planes $\Pi_{2,2}$ are the *twenty faces* of the *five inscribed pyramids* R_2 [28.], of which the *edges* have been seen to be the *thirty lines* $\Lambda_{2,2}$. On the whole, then, there are only *thirty-five planes* Π_2 of *second construction*; which thus divide themselves into *two groups*, of *fifteen* and *twenty*, respectively.

[35.] To *verify arithmetically* (comp. [27.] [28.]) the *completeness* of the foregoing *enumeration* of the *planes* Π_2, we may proceed as follows. In general *fifteen independent points* would determine 455 planes, by their *ternary combinations*; but the 25 *collineations* [28.], which give only the *lines* Λ_1, $\Lambda_{2,1}$ *account* for 25 such combinations, leaving only 430 to be accounted for, by so many *triangles.* Now each plane Π_1 contains three points P_0, and four points P_1, connected by six collineations; it contains therefore 29 (= 35 − 6) triangles, and thus the ten planes Π_1 account for 290 triangles, leaving only 140, situated in planes Π_2. But each of the 15 planes $\Pi_{2,1}$ contains one point P_0, and four points P_1, connected by two collineations; it contains therefore 8 (= 10 − 2) triangles, and thus 120 are accounted for, leaving only 20 ternary combinations to be represented, by triangles in other planes Π_2. And these accordingly have presented themselves, as the twenty faces $\Pi_{2,2}$ of the five inscribed pyramids R_2. It must be mentioned, that the *enumeration* and *classification* of the foregoing *lines* and *planes* had been completely performed by MÖBIUS, although with an entirely different *notation* and *analysis.*

[36.] It is much more difficult, however, or at least without the aid of *types* it *would* be so, to *enumerate* and *classify* what we have called in [1.] the *Points* P_2 of *Second Construction*; and to assign their chief *geometrical relations*, to each other, and to the *five given* and *ten* (formerly) *derived* points, P_0 and P_1. In fact, it is obvious that these *new points* P_2, being (by their definition) *all the intersections* of lines Λ_1 or Λ_2 with planes Π_1 or Π_2, which have *not already occurred*, as points P_0 or P_1, may be expected to be (comp. [2.]) considerably *more numerous*, than either the *lines* or the *planes* themselves.

[37.] The *total number* of *derived lines and planes*, so far, is exactly *one hundred*; namely 55 lines Λ, and 45 planes Π, of first and second constructions. Their *binary combinations*, of the form $\Lambda\Pi$, are therefore 2475 in number; but as it is not difficult to prove that there are 240 distinct cases of *coincidence* of line with plane (or of a plane *containing* a line), we must subtract this from the former number, and thus there remain only 2235 cases of *intersection*, of the kind which we have proposed to consider. *Every one*, however, of these 2235 cases, must be accounted for, either as a *given point* P_0, or as a *derived point* P_1 of *first* construction, or finally as one of those *new points* P_2, of which we have proposed to accomplish the *enumeration*, and to determine the natural *groups*, as represented by their respective *types*.

[38.] We saw, in [27.], that each point P_0, as for instance the point A, represents *twelve intersections* of the form $\Lambda_1 \cdot \Pi_1$: and it is easy to prove that the same point P_0 represents *twelve other* intersections of the form $\Lambda_1 \cdot \Pi_{2,1}$; *twelve* of the form $\Lambda_{2,1} \cdot \Pi_1$; and *three*, of the form $\Lambda_{2,1} \cdot \Pi_{2,1}$; but none of any other form. It represents therefore, in the whole, a system of 39 *intersections*, included in the *general form* $\Lambda \cdot \Pi$; and we must, for this reason, subtract 195 (= 5 × 39) from 2235, leaving 2040 *other* cases of intersection of line with plane, to be accounted for by the old and new *derived points*, P_1 and P_2.

[39.] An analysis of the same kind shows, that each of the *ten points* of *first construction*, as for example the *typical point* A′ [25.], represents *one* intersection of the form $\Lambda_1 \cdot \Pi_1$; *six*, of the form, $\Lambda_1 \cdot \Pi_{2,1}$; *six*, of the form $\Lambda_1 \cdot \Pi_{2,2}$; *six*, of the form $\Lambda_{2,1} \cdot \Pi_1$; *twelve*, of the form $\Lambda_{2,1} \cdot \Pi_{2,1}$; *eighteen*, of the form $\Lambda_{2,1} \cdot \Pi_{2,2}$; *eighteen*, of the form $\Lambda_{2,2} \cdot \Pi_1$; *twenty-four*, of the form $\Lambda_{2,2} \cdot \Pi_{2,1}$; and *twenty-four* others, of the remaining form $\Lambda_{2,2} \cdot \Pi_{2,2}$. It represents, therefore, in all, 115 intersections $\Lambda \cdot \Pi$; and there remain only 890 (= 2040 − 1150) cases of intersection to be accounted for, or represented, by the points P_2 of which we are in search. But all these 890 cases of intersection *must* be accounted for by *such new points*, if the investigation is to be considered as *complete*.

[40.] A *first*, but important, and well-known *group* of such points P_2, consists of the *ten points* (already considered in Part I. of this Paper),

$$\text{A}'', \text{ \&c.}; \quad \text{A}'_2, \text{ \&c.}; \quad \text{A}'_1, \text{ \&c.}; \quad \text{and} \quad \text{D}'_1,$$

namely, the *harmonic conjugates* of the *ten points* P_1, with respect to the *ten lines* Λ_1, which we shall call collectively the points, or the group, $\text{P}_{2,1}$; and among which we shall select the point

$$\text{A}'' = (0\,1\,\overline{1}),$$

as a *Third Typical Point* of the *Net.* In fact, it is what we have called a point P_2, because, without belonging to either of the two former groups P_0 P_1, it is an *intersection* $\Lambda_1 \cdot \Pi_{2,2}$; or rather, it represents *six* such intersections, of the line BC with planes of second construction, and of the second group: namely, with two such through B′C′, two through B_2C_2, and two through B_1C_1, being pairs of faces [28.] of three pyramids R_2, inscribed in those three pyramids R_1, which have been distinguished, in [26.] by the letters A, D, E. The same point A″ is also the intersection of the same line BC with *three planes* $\Pi_{2,1}$; namely, with the three which connect, two by two, the three lines B′C′, B_2C_2, B_1C_1, and contain the three points A, D, E. It is also, in *six* ways, the intersection of one or other of these three last lines $\Lambda_{2,2}$ with a plane Π_1; in *three* ways, with a plane $\Pi_{2,1}$; and in *twelve* ways, with a plane $\Pi_{2,2}$; so that a *single point* $\text{P}_{2,1}$ represents *thirty intersections* of the form $\Lambda \cdot \Pi$; and the group of the *ten* such points represents 300 such intersections. We have therefore only to account for 590 (= 890 − 300) intersections $\Lambda \cdot \Pi$, by *other groups* $\text{P}_{2,2}$, &c., of points of *second construction.*

[41.] A *second group*, $\text{P}_{2,2}$ of such points P_2 has already presented itself, in the case of the *traces* A_0, B_0, C_0 [18.] of the *lines* A_1A_2, B_1B_2, C_1C_2, on the plane ABC. The *ternary* symbol of the point A_0 has been found (77) (92) to be $(\bar{1}\,1\,1)$, its *quinary* symbol is therefore $(\bar{1}\,1\,1\,0\,0)$, which is *congruent* (10) with $(2\,0\,0\,1\,1)$; hence in the *full*, or *quinary sense* [9.], this point A_0 is *syntypical* with the following *other point, in the same plane* ABC,

$$\text{A}''' = (2\,1\,1),$$

which we shall call a *Fourth Typical Point*, and shall consider as representing the *group* $\text{P}_{2,2}$; this group consisting of *thirty* such *points*, namely of two on each of the 15 lines $\Lambda_{2,1}$.

[42.] Each of these thirty points $\text{P}_{2,2}$ represents *seven intersections* of line with plane; namely two of each of the three forms $\Lambda_{2,1} \cdot \Pi_{2,1}$, $\Lambda_{2,1} \cdot \Pi_{2,2}$, $\Lambda_{2,2} \cdot \Pi_{2,1}$, and one of the form $\Lambda_{2,2} \cdot \Pi_1$. For example, the typical point A''', which is the intersection of the *two lines* $\text{AA}'\text{D}_1$ and $\text{B}'\text{C}'$, is at the same time the intersection of the former line $\Lambda_{2,1}$ with each of the four planes Π_2 which contain the latter line $\Lambda_{2,2}$; being also the intersection of this last line $\text{B}'\text{C}'$ with a plane Π_1, namely ADE, and with two planes $\Pi_{2,1}$ which contain the first line $\text{AA}'\text{D}_1$. The *group* $\text{P}_{2,2}$ represents therefore 210 intersections $\Lambda \cdot \Pi$; and there remain only 380 $(= 590 - 210)$ intersections of this standard form, to be accounted for by *other groups* of *second construction*, such as $\text{P}_{2,3}$ &c.

[43.] In investigating such *groups*, we need only seek for *typical points*; and because every such *point* is on a *line* of one of the *three forms* Λ_1, $\Lambda_{2,1}$, $\Lambda_{2,2}$, we may confine ourselves to the *three typical lines*,

$$\text{BC}, \quad \text{AA}'\text{D}_1, \quad \text{B}'\text{C}'; \quad \text{or} \quad (0\,t\,u), \quad (t\,u\,u), \quad (\sigma\,t\,u);$$

in which, as before, $\sigma = t + u$, and in which the ratio of t to u is to be determined. And because a line in the plane ABC intersects any *other plane* in the point in which it intersects the *line* which is the *trace* of the latter plane upon the former, we need only, for the present purpose, consider these lines, or traces: whereof there are, by what has been already seen, *seven* distinct *ternary types*, namely the following:

$$[1\,0\,0], \quad [0\,1\,\bar{1}], \quad [\bar{1}\,1\,1], \quad [1\,1\,1], \quad [0\,1\,1], \quad [\bar{2}\,1\,1], \quad [\bar{2}\,1\,\bar{1}];$$

which answer to the *seven typical traces* of planes,

$$\text{BC}, \quad \text{AA}'\text{D}_1, \quad \text{B}'\text{C}', \quad \text{A}''\text{B}''\text{C}'', \quad \text{AA}'', \quad \text{D}_1\text{A}'', \quad \text{A}'\text{C}_0.$$

There are 22 $(= 3 + 3 + 3 + 1 + 3 + 3 + 6)$ such *lines*, answering to 44 $(= 3\,.\,2 + 3\,.\,3 + 3\,.\,4 + 1\,.\,2 + 3\,.\,1 + 3\,.\,2 + 6\,.\,1)$ *planes*; namely to *all* the 45 planes Π_1, Π_2, *except* the particular plane ABC, on which the *traces* are thus taken. And we have now to *combine* these *seven types of lines*, with the *three symbols of points*, $(0\,t\,u)$, $(t\,u\,u)$, $(\sigma\,t\,u)$, according to the general law $lx + my + nz = 0$ (76).

[44.] The line BC is itself one of the three traces of the first type; and it intersects the twelve other traces, of the five first types, only in points which have been already considered. The line $\mathrm{AA'D_1}$ is, in like manner, a trace of the second type; and it gives no new point, by its intersections with the eight other traces, of the three first types; but its intersection with the common trace $\mathrm{A''B''C''}$, of the two planes $\mathrm{A_1B_1C_1}$ and $\mathrm{A_2B_2C_2}$ [19.], which is the only line of the fourth type, gives what we shall call a *Fifth Typical Point*, namely the following:

$$\mathrm{A^{IV}} = (\overline{2}\,1\,1); \text{ or more fully, } \mathrm{A^{IV}} = (\overline{2}\,1\,1\,0\,0) \equiv (3\,0\,0\,1\,1).$$

This last quinary symbol shows that the point $\mathrm{A^{IV}}$ is syntypical with this other point in the plane ABC,

$$\mathrm{A_1^{IV}} = (3\,1\,1\,0\,0) = (3\,1\,1);$$

so that this *plane* contains *six points* $\mathrm{P}_{2,3}$, which (in the *quinary* sense) belong to one *common group*, although their two *ternary types* are *different*. In fact, the point $\mathrm{A_1^{IV}}$ is the common intersection of the line $\mathrm{AA'D_1}$ with the two planes $[1\,\overline{2}\,\overline{1}\,1\,1]$ and $[1\,\overline{1}\,\overline{2}\,1\,1]$, or $\mathrm{B'C_1C_2}$ and $\mathrm{C'B_1B_2}$, as the point $\mathrm{A^{IV}}$ is the common intersection of the same line with the two planes $[1\,1\,1\,\overline{2}\,\overline{1}]$ $[1\,1\,1\,\overline{1}\,\overline{2}]$, or $\mathrm{A_1B_1C_1}$ and $\mathrm{A_2B_2C_2}$, as above.

[45.] There are *thirty* distinct points $\mathrm{P}_{2,3}$, of this *third group* of *second construction*; and *each* represents *two* (but only two) intersections, which are both of the form $\Lambda_{2,1}\cdot\Pi_{2,2}$. The *group* therefore represents a system of 60 intersections $\Lambda\cdot\Pi$; and there remain only 320 (= 380 − 60) such intersections to be accounted for by *other* points, or groups, such as $\mathrm{P}_{2,4}$, &c. It will be found that we have now exhausted all the points, or groups, of *second* construction, which are situated on lines $\Lambda_{2,1}$; but that two other groups of points P_2 may be determined on lines Λ_1, by combining the typical line BC with the two last sets of traces [43.] as follows.

[46.] Combining thus BC with $\mathrm{D_1C''}$ and $\mathrm{D_1B''}$, or with the traces $[1\,1\,\overline{2}]$ and $[1\,\overline{2}\,1]$, we get the two following points, of a *fourth group of second construction*,

$$\mathrm{A^{V}} = (0\,2\,1); \quad \mathrm{A_1^{V}} = (0\,1\,2);$$

whereof the former may be taken as a *Sixth Typical Point*. There are *twenty points* of this group $\mathrm{P}_{2,4}$, whereof each represents *three* intersections, of the form $\Lambda_1\cdot\Pi_{2,2}$; for example, the typical point $\mathrm{A^V}$ is the common intersection of the line BC with the three planes $\mathrm{C'A_1A_2}$, $\mathrm{D_1A_1B_1}$, $\mathrm{D_1A_2B_2}$; the group therefore represents *sixty* intersections $\Lambda\cdot\Pi$, and there remain 260 (= 320 − 60) to be accounted for.

[47.] Again, combining BC with $\mathrm{C'B_0}$, and with $\mathrm{B'C_0}$, or with $[1\,\overline{1}\,\overline{2}]$ and $[1\,\overline{2}\,\overline{1}]$, we get the two following other points, belonging to a *fifth group* of *second construction*,

$$\mathrm{A^{VI}} = (0\,2\,\overline{1}); \quad \mathrm{A_1^{VI}} = (0\,\overline{1}\,2);$$

whereof the first may be said to be a *Seventh Typical Point*. There are *twenty* points of this new group $\mathrm{P}_{2,5}$, whereof each represents only *one* intersection, of the form $\Lambda_1\cdot\Pi_{2,2}$; for example, $\mathrm{A^{VI}} = \mathrm{BC}\cdot\mathrm{C'B_1B_2}$. We are therefore to subtract 20 from the recent number 260; and thus there remain still 240 intersections to be accounted for, by new points P_2 upon the lines $\Lambda_{2,2}$; since the lines Λ_1 as well as $\Lambda_{2,1}$ have been exhausted, as on examination will easily appear.

[48.] The line $B'C'$ intersects the traces BB'' and CC'' of the *fifth type* [43.] in the two following points, of a *sixth group* of *second construction*,

$$A^{VII} = (1\,2\,\bar{1}); \quad A_1^{VII} = (1\,\bar{1}\,2);$$

whereof the former may be called an *Eighth Typical Point.* There are *sixty* points of this new group, $P_{2,6}$, whereof each represents *one* intersection, of the form $\Lambda_{2,2} \cdot \Pi_{2,1}$; for example A^{VII} is the intersection of the line $B'C'$ with the plane $BC_1A_2A_1C_2$; there remain therefore 180 (= 240 − 60) intersections $\Lambda \cdot \Pi$ to be still accounted for, by other points P_2, on the same set of lines $\Lambda_{2,2}$.

[49.] The traces D_1B'', D_1C'', which belong to the *sixth type* [43.] intersect the line $B'C'$ in two new points, namely

$$A^{VIII} = (3\,2\,1); \quad A_1^{VIII} = (3\,1\,2);$$

which belong to a *seventh group* $P_{2,7}$, of *second construction*, and of which the former may be regarded as a *Ninth Typical Point.* There are *sixty* points of this group, namely two on each of the 30 lines $\Lambda_{2,2}$; and each is the intersection of *one* such line with *two* distinct planes $\Pi_{2,2}$; their *group* therefore represents a system of 120 such intersections; and only 60 (= 180 − 120) intersections *remain* to be accounted for, by *other* points of this last *form*, $\Lambda_{2,2} \cdot \Pi_{2,2}$.

[50.] Accordingly, when we combine the line $B'C'$ with the traces $A'C_0$, $A'B_0$, which are of the *seventh type* [43.], we obtain, for the intersections of that line $\Lambda_{2,2}$ with two new planes $\Pi_{2,2}$, namely with $A'C_1C_2$ and $A'B_1B_2$ (98), two new points, belonging to a new or *eighth group* $P_{2,8}$ of *second construction*, namely,

$$A^{IX} = (2\,3\,\bar{1}); \quad A_1^{IX} = (2\,\bar{1}\,3);$$

whereof the former may be selected, as a *Tenth* (and, for our purpose, *last*) *Typical Point*: for the *sixty* points of this last group represent each *one* intersection, and thus account for *all* the intersections which lately *remained* [49.], after all the preceding groups had been exhausted.

[51.] We are now therefore enabled to assert that the proposed *Enumeration of the Points* P_2 *of Second Construction*, and the proposed *Classification of such Points in Groups*, have both been completely effected. For the *number* of such *groups* $P_{2,1}, \ldots P_{2,8}$ has been seen to be *eight*, represented by the 8 *typical points*, $A'' \ldots A^{IX}$; which, along with the *first given point* A, and the *first derived point* A', make up a system of *ten types*, as follows:

$$A = (1\,0\,0); \quad A' = (0\,1\,1); \quad A'' = (0\,1\,\bar{1}); \quad A''' = (2\,1\,1); \quad A^{IV} = (\bar{2}\,1\,1);$$

$$A^{V} = (0\,2\,1); \quad A^{VI} = (0\,2\,\bar{1}); \quad A^{VII} = (1\,2\,\bar{1}); \quad A^{VIII} = (3\,2\,1); \quad A^{IX} = (2\,3\,\bar{1});$$

and the *number* of the *points* P_2 is (10 + 30 + 30 + 20 + 20 + 60 + 60 + 60 =) 290; so that, when combined with the points P_1, they make up a system of exactly *three hundred points,* P_1*,* P_2*, derived from the five points* P_0.

[52.] It is to be remembered that the three other *ternary types*,

$$\text{D}_1 = (1\,1\,1), \quad \text{A}_0 = (\bar{1}\,1\,1), \quad \text{A}_1^{\text{IV}} = (3\,1\,1),$$

have been seen to represent points which are, in the *quinary* sense, *syntypical* with A', A''', A^{IV}, and therefore belong to the same three groups, P_1, $\text{P}_{2,2}$, $\text{P}_{2,3}$; all these three points being in the plane ABC, and on the line $\text{AA}'\text{D}_1$. And it is evident that the five other points,

$$\text{A}_1^{\text{V}} = (0\,1\,2); \quad \text{A}_1^{\text{VI}} = (0\,\bar{1}\,2); \quad \text{A}_1^{\text{VII}} = (1\,\bar{1}\,2); \quad \text{A}_1^{\text{VIII}} = (3\,1\,2); \quad \text{A}_1^{\text{IX}} = (2\,\bar{1}\,3),$$

belong (as has been seen) to the same five last groups $\text{P}_{2,4}, \dots \text{P}_{2,8}$, as the five points above selected as typical thereof, namely the points $\text{A}^{\text{V}} \dots \text{A}^{\text{IX}}$, and are situated on the same two typical lines BC and $\text{B}'\text{C}'$. The transition from A' to B', C', or from A'' to B'', C'', &c., is very easily made, by a rule already stated [20.]; and therefore it is unnecessary to write down here the symbols for *these* derived points, B', B'', &c., or C', C'' &c. But we must now proceed, in the remainder of this Paper, to investigate some of the chief *Geometrical Relations* which connect the points, lines, and planes of the *Net*, so far as they have been hitherto determined: namely to the end of the *Second Construction*.

Part III.—*Applications to the Net, continued: Enumeration and Classification of the Collineations of the Fifty-Two Points in a Plane of First Construction.*

[53.] The plane ABC has been seen to contain, besides the three points P_0 which determine it, four points P_1, namely A', B', C', and D_1; and it contains forty-five points P_2, namely the three points A'', B'', C'' of the group $\text{P}_{2,1}$, and six points of each of the seven remaining groups of second construction. This *plane* Π_1 contains therefore *fifty-two points* P_0, P_1, P_2; and we propose to examine, in the first place, the various *relations of collinearity* which connect these different points among themselves: intending afterwards to investigate their principal *harmonic and involutionary relations.*

[54.] The points on the *first typical line* BC [23.] are, in number, *eight*; their literal symbols being, by what precedes,

$$\text{B}, \ \text{C}, \ \text{A}', \ \text{A}'', \ \text{A}^{\text{V}}, \ \text{A}_1^{\text{V}}, \ \text{A}^{\text{VI}}, \ \text{A}_1^{\text{VI}};$$

the ternary symbols corresponding to which have been shown to be,

$$(0\,1\,0), \quad (0\,0\,1), \quad (0\,1\,1), \quad (0\,1\,\bar{1}), \quad (0\,2\,1), \quad (0\,1\,2), \quad (0\,2\,\bar{1}), \quad (0\,\bar{1}\,2).$$

In fact, that these eight points are all on the line BC, is evident on mere inspection of their *symbols*, which are of the common *form*,

$$(0\,y\,z) \quad [23.].$$

[55.] The points on the *second typical line*, AA' [30.], are in number *seven*: their literal symbols being,

$$\text{A}, \ \text{A}', \ \text{D}_1, \ \text{A}''', \ \text{A}_0, \ \text{A}^{\text{IV}}, \ \text{A}_1^{\text{IV}};$$

and their ternary symbols being,

$$(1\,0\,0), \quad (0\,1\,1), \quad (1\,1\,1), \quad (2\,1\,1), \quad (\bar{1}\,1\,1), \quad (\bar{2}\,1\,1), \quad (3\,1\,1).$$

In fact, each of these seven symbols is evidently of the form $(t\,u\,u)$, or $(x\,y\,y)$ [30.].

[56.] The points on the *third typical line*, B′C′ [31.], are in number *ten*; namely the points,

$$\text{B}', \ \text{C}', \ \text{A}'', \ \text{A}''', \ \text{A}^{\text{VII}}, \ \text{A}_1^{\text{VII}}, \ \text{A}^{\text{VIII}}, \ \text{A}_1^{\text{VIII}}, \ \text{A}^{\text{IX}}, \ \text{A}_1^{\text{IX}},$$

of which the ternary symbols are,

$$(1\,0\,1), \quad (1\,1\,0), \quad (0\,1\,\overline{1}), \quad (2\,1\,1), \quad (1\,2\,\overline{1}), \quad (1\,\overline{1}\,2), \quad (3\,2\,1), \quad (3\,1\,2), \quad (2\,3\,\overline{1}), \quad (2\,\overline{1}\,3);$$

each of these ten symbols being of the form $(\sigma\,t\,u)$ [31.], with $\sigma = t + u$, as before.

[57.] These *three* typical *lines*, in the plane ABC, which may be denoted by the ternary symbols, $[1\,0\,0]$, $[0\,1\,\overline{1}]$, $[\overline{1}\,1\,1]$, and represent a system of *nine lines* Λ_1, Λ_2 in that plane Π_1, are also three typical *traces* [43.] of *other* planes thereon; and the remaining traces of such planes are in number *thirteen*, represented by *four* other lines, as *types*: of which lines, considered as such traces, the ternary symbols have been found [43.] to be,

$$[1\,1\,1], \quad [0\,1\,1], \quad [\overline{2}\,1\,1], \quad [\overline{2}\,1\,\overline{1}];$$

answering to the literal symbols,

$$\text{A}''\text{B}''\text{C}'', \quad \text{AA}'', \quad \text{D}_1\text{A}'', \quad \text{A}'\text{C}_0,$$

and serving as abridged expressions for the four *equations* of ternary form,

$$x + y + z = 0, \quad y + z = 0, \quad 2x = y + z, \quad 2x = y - z.$$

[58.] Each of these four last lines passes through *six* points; thus the trace $[1\,1\,1]$ passes through the points $(0\,1\,\overline{1})$ $(\overline{1}\,0\,1)$ $(1\,\overline{1}\,0)$ $(\overline{2}\,1\,1)$ $(1\,\overline{2}\,1)$ $(1\,1\,\overline{2})$, or through $\text{A}''\,\text{B}''\,\text{C}''\,\text{A}^{\text{IV}}\,\text{B}^{\text{IV}}\,\text{C}^{\text{IV}}$; $[0\,1\,1]$ through $(1\,0\,0)$ $(0\,1\,\overline{1})$ $(1\,\overline{1}\,1)$ $(1\,1\,\overline{1})$ $(2\,\overline{1}\,1)$ $(2\,1\,\overline{1})$, or $\text{A}\,\text{A}''\,\text{B}_0\,\text{C}_0\,\text{C}^{\text{VII}}\,\text{B}_1^{\text{VII}}$; $[\overline{2}\,1\,1]$ through $(1\,1\,1)$ $(0\,1\,\overline{1})$ $(1\,0\,2)$ $(1\,2\,0)$ $(2\,1\,3)$ $(2\,3\,1)$, or $\text{D}_1\,\text{A}''\,\text{B}^{\text{V}}\,\text{C}_1^{\text{V}}\,\text{C}^{\text{VIII}}\,\text{B}_1^{\text{VIII}}$; and $[\overline{2}\,1\,\overline{1}]$ through $(0\,1\,1)$ $(1\,1\,\overline{1})$ $(1\,3\,1)$ $(1\,2\,0)$ $(\overline{1}\,0\,2)$ $(2\,3\,\overline{1})$, or $\text{A}'\,\text{C}_0\,\text{B}_1^{\text{IV}}\,\text{C}_1^{\text{V}}\,\text{B}^{\text{VI}}\,\text{A}^{\text{IX}}$; the correctness of the *ternary symbols* being evident on inspection, if the law $lx + my + nz = 0$ (76) be remembered: and the *literal symbols* being thence at once deduced, by [51.] and [52.].

[59.] *So far*, then, that is when we attend only to the *twenty-two traces* [43.] of planes Π_1, Π_2 on the plane ABC, we find a system of three collineations of eight points; three of seven points; three of ten points; and thirteen of six points each. Each collineation of the first of these four systems *counts* as 28 binary combinations of the 52 points in the plane [53.]; each of the second system counts as 21 such combinations; each of the third system as 45; and each of the fourth as 15. We therefore account, in this way, for $84 + 63 + 135 + 195 = 477$ binary combinations; but the total number is $26 \,.\, 51 = 1326$; there remain then 849 to be accounted for, by lines Λ_3 which are *not traces*, of any one of the foregoing groups.

[60.] In seeking for such new lines, it is natural to consider first those which pass through one or other of the three given points A, B, C; and the *types* of such are found to be the five following, each representing a new group of six lines Λ_3:

$$[0\,2\,1];\quad [0\,2\,\bar{1}];\quad [0\,3\,\bar{1}];\quad [0\,3\,\bar{2}];\quad [0\,3\,1].$$

As *symbols*, these answer respectively to the five *new lines*:

$$\begin{array}{ll}
(1\,0\,0)\ (1\,1\,\bar{2})\ (0\,\bar{1}\,2)\ (1\,\bar{1}\,2)\ (3\,\bar{1}\,2), & \text{or } \mathrm{A}\,\mathrm{C}^{\mathrm{IV}}\,\mathrm{A}_1^{\mathrm{VI}}\,\mathrm{A}_1^{\mathrm{VII}}\,\mathrm{C}^{\mathrm{IX}};\\
(1\,0\,0)\ (1\,1\,2)\ (0\,1\,2)\ (\bar{1}\,1\,2)\ (3\,1\,2), & \text{or } \mathrm{A}\,\mathrm{C}'''\,\mathrm{A}_1^{\mathrm{V}}\,\mathrm{B}^{\mathrm{VII}}\,\mathrm{A}_1^{\mathrm{VIII}};\\
(1\,0\,0)\ (1\,1\,3)\ (2\,1\,3), & \text{or } \mathrm{A}\,\mathrm{C}_1^{\mathrm{IV}}\,\mathrm{C}^{\mathrm{VIII}};\\
(1\,0\,0)\ (1\,2\,3)\ (\bar{1}\,2\,3), & \text{or } \mathrm{A}\,\mathrm{C}_1^{\mathrm{VIII}}\,\mathrm{B}^{\mathrm{IX}};\\
(1\,0\,0)\ (2\,\bar{1}\,3), & \text{or } \mathrm{A}\,\mathrm{A}_1^{\mathrm{IX}}.
\end{array}$$

We have thus *twelve* lines Λ_3, each connecting a point P_0, with *four* points P_2, and counting as *ten* binary combinations; *twelve* other lines, each connecting a point P_0 with *two* points P_2, and counting as *three* such combinations; and *six* lines, each of which connects a point P_0 with *one* point P_2, and counts as only *one* combination. In this manner, then, we account for $120 + 36 + 6 = 162$, out of the 849 which had remained in [59.]; but there still remain 687 combinations to be accounted for, by new lines of third construction, which pass through no given point.

[61.] Considering next the new lines which connect a point of the *first* construction, with one or more points of the *second*, we find these five new types,

$$[3\,1\,\bar{1}];\quad [1\,2\,\bar{2}];\quad [1\,2\,\bar{3}];\quad [1\,3\,\bar{3}];\quad [1\,3\,\bar{4}];$$

which as symbols denote the five lines,

$$\left.\begin{array}{r}
(0\,1\,1)\ (1\,\bar{2}\,1)\ (1\,\bar{1}\,2);\quad (0\,1\,1)\ (2\,0\,1)\ (2\,\bar{1}\,0);\quad (1\,1\,1)\ (2\,\bar{1}\,0)\ (\bar{1}\,2\,1);\\
(0\,1\,1)\ (3\,1\,2);\quad (1\,1\,1)\ (\bar{1}\,3\,2);
\end{array}\right\}$$

$$\text{or } \mathrm{A}'\,\mathrm{B}^{\mathrm{IV}}\,\mathrm{A}_1^{\mathrm{VII}};\ \mathrm{A}'\,\mathrm{B}_1^{\mathrm{V}}\,\mathrm{C}^{\mathrm{VI}};\ \mathrm{D}_1\,\mathrm{C}^{\mathrm{VI}}\,\mathrm{C}_1^{\mathrm{VII}};\ \mathrm{A}'\,\mathrm{A}_1^{\mathrm{VIII}};\ \text{and } \mathrm{D}_1\,\mathrm{C}_1^{\mathrm{IX}};$$

but as *types* represent each a *group of six lines*. We thus get 18 new lines, each passing through 1 point P_1, and 2 points P_2; and 12 other lines, each connecting a point P_1 with only *one* point P_2. And these thirty lines Λ_3 account for $54 + 12 = 66$ binary combinations of points; leaving however 621 such combinations to be accounted for, by new lines Λ_3, of which each must connect at least two points P_2, without passing through any point P_0 or P_1, and without being any one of the traces already considered.

[62.] The *symbol* $[\bar{2}\,3\,3]$, which denotes a line passing through *two* points P_2, namely, $(0\,1\,\bar{1})$ and $(3\,1\,1)$, or A'' and $\mathrm{A}_1^{\mathrm{IV}}$, but through *no other* point, represents, when considered as a *type*, a group of *three* such lines; and 40 *other types*, as for example $[1\,\bar{3}\,4]$, which as a symbol denotes the line $(\bar{1}\,1\,1)$ $(1\,3\,2)$, or $\mathrm{A}_0\,\mathrm{B}^{\mathrm{VIII}}$, are found to exist, representing each a group of *six* lines, whereof each connects in like manner *two* points P_2, but *only* those two points. We have thus a system of 243 new lines, which represent only so many binary combinations: and there remain 378 such combinations to be accounted for, by new lines Λ_3, whereof each must connect *at least three points* P_2.

[63.] For lines connecting *three* such points, and *no more*, it is found that there are *twenty types*; whereof *eight*, as for instance the type $[\overline{3}\,1\,1]$, which as a symbol denotes the line $(0\,1\,\overline{1})$ $(1\,2\,1)$ $(1\,1\,2)$, or $\text{A}''\,\text{B}'''\,\text{C}'''$, represent each a group of *three* such lines; while each of the *twelve others*, like $[1\,\overline{2}\,3]$, which as a symbol denotes the line $(\overline{1}\,1\,1)$ $(1\,2\,1)$ $(2\,1\,0)$, or $\text{A}_0\,\text{B}'''\,\text{C}^{\text{V}}$, represents a group of *six* lines. We have thus 96 new lines, counting as 288 binary combinations: but we must still account for 90 *other* combinations, by new lines Λ_3, connecting each *more than three points* P_2.

[64.] Accordingly, we find *three new types of lines*, which *alone remain*, when all those which have been above exhibited, or alluded* to are set aside: namely

$$[1\,\overline{2}\,4]; \quad [\overline{1}\,2\,4]; \quad [1\,1\,2].$$

And those represent, respectively, groups of *six*, of *six*, and of *three* new lines, and therefore on the whole a system of *fifteen* new lines, each passing *through four points* P_2, and consequently counting as *six* combinations; for example, as *symbols*, they denote the three following lines:

$$\begin{array}{ll} (2\,1\,0)\ (\overline{2}\,1\,1)\ (0\,2\,1)\ (2\,3\,1), & \text{or } \text{C}^{\text{V}}\,\text{A}^{\text{IV}}\,\text{A}^{\text{V}}\,\text{B}_1^{\text{VIII}}; \\ (2\,1\,0)\ (2\,\overline{1}\,1)\ (0\,2\,\overline{1})\ (2\,3\,\overline{1}), & \text{or } \text{C}^{\text{V}}\,\text{C}^{\text{VII}}\,\text{A}^{\text{VI}}\,\text{A}^{\text{IX}}; \\ (2\,0\,\overline{1})\ (1\,\overline{1}\,0)\ (0\,2\,\overline{1})\ (1\,1\,\overline{1}), & \text{or } \text{B}_1^{\text{VI}}\,\text{C}''\,\text{A}^{\text{VI}}\,\text{C}_0. \end{array}$$

But $6 \,.\, 15 = 90$; we are therefore entitled to say, that *all the* 1326 *binary combinations* [59.], *of the* 52 *points* P_0, P_1, P_2 [53.] *in the plane* ABC, *have now been fully accounted for.*

[65.] Collecting the results, respecting the *collineations in the plane* ABC, it has been found that there are 261 *lines* Λ_3, whereof each *connects two*, but *only two*, of the the 52 *points* in that plane; and that *these lines*, which at the present stage of the construction are not properly *cases of collinearity* at all, are represented by a system of 44 *ternary types.*

[66.] There are 126 *other lines* Λ_3, each connecting *three* (but *only three*) *points*; they are represented by a system of 25 *types*; and account for 378 binary *combinations.*

[67.] There are 15 lines Λ_3, each connecting *four points* P_2; they are represented by a system of 3 types, and account for 90 combinations.

[68.] There are 12 lines Λ_3, each connecting *one* point P_0 with *four* points P_2; they are represented by 2 types, and represent 120 combinations.

[69.] There are 13 other lines Λ_3, namely the *traces of planes* Π_1 or Π_2, whereof each connects *six points*, namely a point P_0 or P_1 with five points P_2, or else six points P_2 with each other; they are represented by 4 types, and account for 195 combinations.

* It has been thought that it could not be interesting to set down *all* the *types of lines*, above referred to; especially as those which relate to lines *not* passing through *at least four points* give rise, at the present stage of the construction, to no *theorems* of *harmonic* (or *anharmonic*) *ratio.*

[70.] There are 3 lines $\Lambda_{2,2}$, each connecting *two* points P_1 with *eight* points P_2; they have one common type, and represent 135 combinations.

[71.] There are, in like manner, 3 lines $\Lambda_{2,1}$, each connecting *one point* P_0 with *two points* P_1, and with *four points* P_2, but having only *one* common type; and they represent 63 combinations.

[72.] Finally, there are (in the same plane) 3 *lines* Λ_1, each connecting *two points* P_0 with *one point* P_1, and with *five points* P_2; these lines also have all *one type*; and they account for 84 *combinations*: with the *arithmetical verification*, that

$$261 + 378 + 90 + 120 + 195 + 135 + 63 + 84 = 1326 = 26 \,.\, 51;$$

which proves that the *enumeration* is *complete*.

[73.] The *total number of distinct lines*, above obtained, is $261 + 126 + 15 + 12 + 13 + 3 + 3 + 3 = 436$; and the total number of their *ternary types* is 81. But *if we set aside* (as conducting to *no general metric relations*) *all those lines which contain fewer than four points*, there *then remain only forty-nine lines*, and *only twelve types*, to be discussed, with reference to *harmonic* (or *anharmonic*) *relations*, of the points upon those lines.

[74.] For the purpose of studying completely all *such* relations, it will therefore be permitted to confine ourselves to the *three first typical lines*, BC, AA′, B′C′, or $[1\,0\,0]$, $[0\,1\,\overline{1}]$, $[\overline{1}\,1\,1]$; the *four other typical traces*, A″B″C″, AA″, D_1A″, A′C_0, or $[1\,1\,1]$, $[0\,1\,1]$, $[\overline{2}\,1\,1]$, $[\overline{2}\,1\,\overline{1}]$; and *five new typical lines* Λ_3, connecting each *at least four points*: namely the *two lines*, $[0\,2\,1]$ and $[0\,2\,\overline{1}]$, of [60.], whereof each connects the given point A with *four* points P_2; and the *three lines* $[1\,\overline{2}\,4]$, $[\overline{1}\,2\,4]$, $[1\,1\,2]$, of [64.], of which each connects *four* other points P_2 among themselves, but does not pass through any point P_0 or P_1.

Part IV.—*Applications to the Net, continued: Harmonic and Involutionary Relations, of the Points situated on the Twelve Typical Lines, in a Plane of First Construction.*

[75.] Commencing here with the examination of the last typical lines, because they contain only *four* points each, let us adopt, as temporary symbols, of the *literal* kind, the ten following:

$$\begin{array}{llll} a = (2\,1\,0), & b = (\overline{2}\,1\,1), & c = (0\,2\,1), & d = (2\,3\,1); \\ & b' = (2\,\overline{1}\,1), & c' = (0\,2\,\overline{1}), & d' = (2\,3\,\overline{1}); \\ a'' = (2\,0\,\overline{1}), & b'' = (1\,\overline{1}\,0), & & d'' = (1\,1\,\overline{1}); \end{array}$$

instead of the more systematic but less simple symbols, C^{V} A^{IV} A^{V} B_1^{VIII} C^{VII} A^{VI} A^{IX} B_1^{VI} C″ C_0.

[76.] The three lines referred to [64.], are then the three following:

$$abcd; \quad ab'c'd'; \quad a''b''c'd''.$$

And because we have (comp. [16.]) the six symbolical relations,

$$(c)-(a)=(b); \quad (c)+(a)=(d);$$
$$(a)-(c')=(b'); \quad (a)+(c')=(d');$$
$$(a'')-(c')=2(b''); \quad (a'')+(c')=2(d''),$$

it results (31) that the three *harmonic equations* exist:

$$(a\,b\,c\,d)=(a\,b'\,c'\,d')=(a''\,b''\,c'\,d'')=-1.$$

We have therefore this *Theorem*:—

"*Each of the* 150 *lines* Λ_3*, which connect four points* P_2*, in any one of the ten planes* Π_1*, and pass through no other of the* 305 *points* P_0*,* P_1*,* P_2*, is harmonically divided.*"

[77.] As verifications, the three right lines bb', cc', dd' concur in the point C; bd', cc', db', in B; aa'', $b'b''$, $d'd''$, in A′; and aa'', $b'd''$, $d'b''$, in a point P_3, namely in $(4\,1\,\overline{1})$: the existence of which four *concurrences* of lines was to be expected, from a known principle of *homography*, as consequences of the harmonic relations [76.]. It is worth noticing, however, how simply these concurrences are here *expressed*, by the *ternary symbols* of the *points*, according to the *law* (18); or, if we choose, by the corresponding symbols of the *lines*, with the analogous law (25): for example, the three last concurrent lines, aa'', &c., have for their respective symbols, $[1\,\overline{2}\,2]$, $[0\,1\,1]$, and $[1\,1\,5]=[1\,\overline{2}\,2]+[0\,3\,3]$.

[78.] To examine, in like manner, the analogous relations of arrangement, on the two new typical lines [60.], namely $[0\,2\,1]$ and $[0\,2\,\overline{1}]$, whereof each connects the given point A with four points of second construction, let us write as eight new temporary symbls of the literal kind, more convenient than the former symbols, $\mathrm{C}^{\mathrm{IV}}\ \mathrm{A}_1^{\mathrm{VI}}\ \mathrm{A}_1^{\mathrm{VII}}\ \mathrm{C}^{\mathrm{IX}}\ \mathrm{B}^{\mathrm{VII}}\ \mathrm{A}_1^{\mathrm{V}}\ \mathrm{C}'''\ \mathrm{A}_1^{\mathrm{VIII}}$, the following:

$$b=(1\,1\,\overline{2}), \quad c=(0\,\overline{1}\,2), \quad d=(1\,\overline{1}\,2), \quad e=(3\,\overline{1}\,2);$$
$$\beta=(\overline{1}\,1\,2), \quad \gamma=(0\,1\,2), \quad \delta=(1\,1\,2), \quad \epsilon=(3\,1\,2),$$

so that the two lines in question are,

$$\mathrm{A}bcde, \quad \text{and} \quad \mathrm{A}\beta\gamma\delta\epsilon.$$

We have thus the eight following new symbolical relations, A being still $=(1\,0\,0)$:

$$(\mathrm{A})-(c)=(b), \quad (\mathrm{A})+(c)=(d); \qquad (e)-(b)=2(d), \quad (e)+(b)=4(\mathrm{A});$$
$$(\gamma)-(\mathrm{A})=(\beta), \quad (\gamma)+(\mathrm{A})=(\delta); \qquad (\epsilon)+(\beta)=2(\delta), \quad (\epsilon)-(\beta)=4(\mathrm{A});$$

whence result at once the *four harmonic relations*,

$$(\mathrm{A}\,b\,c\,d)=(\mathrm{A}\,b\,d\,e)=(\mathrm{A}\,\beta\,\gamma\,\delta)=(\mathrm{A}\,\beta\,\delta\,\epsilon)=-1.$$

These *two* lines from A are therefore *homographically divided*, the point A corresponding to *itself*, and b to β, &c.; and accordingly the *four right lines*, $b\beta$, $c\gamma$, $d\delta$, $e\epsilon$, *which connect corresponding points*, concur in one common point, which is easily found to be B. And other *verifications*, by such *concurrences*, can be assigned with little trouble.

[79.] It may assist the conception of the *common law of arrangement*, of the *five points* on each of the *two typical lines* last considered, to suppose that the joining line $b\beta$ is *thrown off*, by projection, *to infinity*; or, what comes to the same thing, that the *two points* b and β, themselves, are thus made infinitely distant. For thus the harmonic equations [78.] will simply express that, *in this projected state of the figure*, the *four points*, d, e, δ, ϵ, *bisect* respectively the *four intervals*, Ac, Ad, Aγ, Aδ; whence it is easy to construct a diagram, not necessary here to be exhibited. The consideration of the *two other lines* through the same given point A, which have $[0\,1\,2]$ $[0\,\overline{1}\,2]$ for their symbols, and belong to the same two types as the two last, would offer to our notice a *pencil of four rays*, which has some interesting properties, especially as regards its *intersections* with *other* pencils, but which we cannot here delay to describe.

[80.] It may, however, be worth while to state here, as a consequence from the preceding discussion, this other *Theorem*:—

"*The* 120 *lines* Λ_3 *in the ten planes* Π_1*, whereof each connects a point* P_0 *with four points* P_2*, and with no other of the* 305 *points, although not all syntypical, are all homographically divided.*"

[81.] Proceeding to consider the arrangements of those six typical lines [58.] which contain each *six points*, we find that whether we write, as new temporary and literal symbols,

$$a=(0\,1\,\overline{1}),\quad b=(\overline{1}\,0\,1),\quad c=(1\,\overline{1}\,0),\quad a'=(\overline{2}\,1\,1),\quad b'=(1\,\overline{2}\,1),\quad c'=(1\,1\,\overline{2}),$$

or

$$a=(0\,1\,1),\quad b=(1\,1\,\overline{1}),\quad c=(1\,2\,0),\quad a'=(2\,3\,\overline{1}),\quad b'=(1\,3\,1),\quad c'=(\overline{1}\,0\,2),$$

the six points $a\,b\,c\,a'\,b'\,c'$ being in the one case on the line $[1\,1\,1]$, and in the other case on the line $[\overline{2}\,1\,\overline{1}]$, we have in each case the three harmonic equations:

$$(c\,a\,b\,a')=(a\,b\,c\,b')=(b\,c\,a\,c')=-1.$$

We may then at once infer this *Theorem*:

"*The* 70 *lines* Λ_3*, in the ten planes* Π_1*, which are represented by the fourth and seventh typical traces of planes on the plane* ABC*, although not all syntypical* (or generated by similar processes of construction), *are all homographically divided.*"

[82.] This *common mode* of their *division* may deserve, however, a somewhat closer examination, its consequences being not without interest. When any six collinear points, $a\,\ldots\,c'$, are connected by the three equations [81.], we are permitted to suppose that their symbols are so *prepared* (if necessary), by *coefficients*,* as to give,

$$(a)+(b)+(c)=0;$$

* For example, in the second case [81.], we should change the symbols for c and b' to their negatives, before employing the formulæ of [82.].

$$(a') = (b) - (c), \quad (b') = (c) - (a), \quad (c') = (a) - (b);$$

and therefore,

$$(a') + (b') + (c') = 0,$$

$$3(a) = (c') - (b'), \quad 3(b) = (a') - (c'), \quad 3(c) = (b') - (a').$$

Whenever, then, the three harmonic equations [81.] *exist, for a system of six collinear points*, $a \ldots c'$, *the three other harmonic equations*, formed by interchanging accented and unaccented letters,

$$(c'\, a'\, b'\, a) = (a'\, b'\, c'\, b) = (b'\, c'\, a'\, c) = -1,$$

are also satisfied; and *the three pairs* (or *segments*),

$$aa', \quad bb', \quad cc',$$

*which connect corresponding points, compose an involution.**

[83.] Under the same conditions, the two points a and a' are harmonically conjugate to each other, not only with respect to b and c, but also with respect to b' and c'; they are therefore the *double points* (or *foci*) of that *other involution* which is determined by the *two pairs* of points, bc, $b'c'$. In like manner, b, b' are the double points of the involution, determined by the two pairs, or segments ca, $c'a'$; and c, c' are the double points of the involution determined by ab, $a'b'$.

[84.] From any one of the three last involutions [83.], we could *return*, by known principles, to the involution [82.]; we can also infer from them that the *three new pairs of points* (or *segments of the common line*), aa', bc', cb'; the three pairs, or segments, bb', ca', ac'; and the three others, cc', ab', ba', form *three other involutions*, making *seven distinct involutions of the six points*, so far: in *three* of which, as we have seen in [83.] *two* of those *six points* are *their own conjugates.*

[85.] For these and other reasons we propose to say, that *when any three collinear points* (as a, b, c) *are assumed* (or *given*), *and three other points on the same line are derived from them, by the condition that each shall be the harmonic conjugate of one, with respect to the other two, then these two sets of points are two Triads of Points in Involution.* And it is easy to extend this definition so as to include cases of two *triads* of complanar and co-initial *lines*, or of collinear *planes*, which shall be, in the same general but (as it is supposed) *new sense*, in *involution* with each other: every such *involution of triads* including, by what precedes, *a system of seven involutions* of the *old* or *usual* kind.

* Compare p. 127 of the *Géométrie Supérieure* (Paris, 1852). In general, the reader is supposed to be acquainted with the chapter (chap. ix.) of that excellent work of M. *Chasles*, which treats of *Involution.*

[86.] For example, because the two *triads of points*, $\text{A}''\text{B}''\text{C}''$ and $\text{A}^{\text{IV}}\text{B}^{\text{IV}}\text{C}^{\text{IV}}$, are thus in involution, by the equations [81.] applied to the fourth typical trace [43.], it follows that the *two pencils*, each of *three rays*,

$$\text{D}_1 \,.\, \text{A}''\text{B}''\text{C}'', \quad \text{and} \quad \text{D}_1 \,.\, \text{ABC},$$

are *triads of lines, in involution* with each other; and that, for a similar reason, the *two triads of planes*, all passing through the line DE,

$$\text{DEA}, \quad \text{DEB}, \quad \text{DEC}, \quad \text{and} \quad \text{DEA}'', \quad \text{DEB}'', \quad \text{DEC}'',$$

are, in the sense above explained, in *involution.* In fact, when the point D_1 is thus taken as a *vertex of the pencils* in the plane ABC, the three harmonic equations of the first case [81.], namely,

$$(\text{C}''\,\text{A}''\,\text{B}''\,\text{A}^{\text{IV}}) = (\text{A}''\,\text{B}''\,\text{C}''\,\text{B}^{\text{IV}}) = (\text{B}''\,\text{C}''\,\text{A}''\,\text{C}^{\text{IV}}) = -1,$$

or rather the three reciprocal equations (comp. [82.]),

$$(\text{C}^{\text{IV}}\,\text{A}^{\text{IV}}\,\text{B}^{\text{IV}}\,\text{A}'') = (\text{A}^{\text{IV}}\,\text{B}^{\text{IV}}\,\text{C}^{\text{IV}}\,\text{B}'') = (\text{B}^{\text{IV}}\,\text{C}^{\text{IV}}\,\text{A}^{\text{IV}}\,\text{C}'') = -1,$$

correspond simply to the elementary equations, (50), (56),

$$(\text{C}\,\text{A}'\,\text{B}\,\text{A}'') = (\text{A}\,\text{B}'\,\text{C}\,\text{B}'') = (\text{B}\,\text{C}'\,\text{A}\,\text{C}'') = -1,$$

which may be employed to *define* the three important points A'', B'', C'', (87), of the *first group* of *second construction* [40.], as being the (well known) *harmonic conjugates* of the points A', B', C' of *first* construction, with respect to the three *lines* of the same first construction, BC, CA, AB, on which those points are situated.

[87.] The equations [82.], which connect the *symbols* $(a) \ldots (c')$ of the *six points*, give, by easy eliminations, these other equations of the same kind:

$$(b') = (b) + 2(c); \quad -(c') = 2(b) + (c);$$

we have therefore, by (31), the following *anharmonic of the group* b, b', c, c':

$$(b\,b'\,c\,c') = +4;$$

and other easy calculations of the same sort given, in like manner, the equal anharmonics,

$$(c\,c'\,a\,a') = +4; \quad (a\,a'\,b\,b') = +4.$$

But in general, for any four collinear points, a, b, c, d, the *definition* (29) of the *symbol* $(a\,b\,c\,d)$ gives easily the relation,

$$(a\,b\,c\,d) + (a\,c\,b\,d) = 1;$$

and hence, or immediately by calculations such as those recently used, we have this *other* set of anharmonics, with a *new* common value:

$$(b\,c\,b'\,c') = (c\,a\,c'\,a') = (a\,b\,a'\,b') = -3;$$

the *negative* character of which shows, by the same definition (29), that the segment (or interval) aa', for example, is cut *internally* by *one* of the two points b, b', or by one of the two points c, c', and *externally* by the *other*; with similar results for each of the two other segments, bb', cc'.

[88.] We may then say that *each of the three segements, aa', bb', cc', overlaps each of the two others*, in the sense that *any two* of them have a *common part*, and also *parts not common*: whence it immediately follows that the *involution* [82.], *to which these three segments belong, has its double points imaginary*: whereas it may be proved, on the same plan, that each of the three involutions of segments mentioned in [84.], namely aa', bc', cb'; bb', ca', ac'; cc', ab', ba', has *real* double points*; and the double points of the three other involutions, determined by the three *pairs* of segments, bc, $b'c'$; ca, $c'a'$; ab, $a'b'$, are likewise *real*, and have been assigned [83.]; namely, in each of these three last cases, the two remaining points of the system.

[89.] Now, in general, when the *foci* (or double points) of an involution of collinear segments, aa', bb',... are *imaginary*, so that *conjugate points*, a, a', or b, b', &c., fall at *opposite sides* of the *central point* O, it is known, and may indeed by considered as evident, that if an *ordinate* OP be erected, equal to the constant *geometrical mean* between the two distances Oa, Oa', or Ob, Ob', &c., then, *all the segments aa', bb', &c., subtend right angles, at the extremity* P *of this ordinate.* It follows, then, by what has been proved in [82.] and [88.], and by the *first case* of [81.], that *each of the three segments* $A''A^{IV}$, $B''B^{IV}$, $C''C^{IV}$, *of the fourth typical trace* [43.] *subtends a right angle at some one point*, P, *in the plane* ABC, or rather generally at *each* of *two* such points: and in like manner, by the *second case* [81.], that each of the *three other segments*, $A'A^{IX}$, $C_0B_1^{IV}$, $C_1^{V}B^{VI}$, of the *seventh typical trace*, subtends a *right angle*, at each of two *other* points, P, P$'$, in the same plane.

[90.] *These* results, by their nature, like *all the foregoing results* of the present Paper, are quite *independent of the assumed arrangement of the five given* (or *initial*) *points of space* A ... E, and are *unaffected by projection*, or *perspective.* In saying this, it is not meant, of course, that one *right angle* will generally be *projected* into *another*; or that the *new point* P, at which the *three new segments* $A''A^{IV}$, $B''B^{IV}$, $C''C^{IV}$, or $A'A^{IX}$, $C_0B_1^{IV}$, $C_1^{V}B^{VI}$, *subtend* right angles, will be itself (what may be called) the *projection* of the *old point* P [89.], which was so related to the three *old segments*, denoted by the same literal symbols, when the *arrangement* (or *configuration*) of the five *initial* points is *varied*, by a process *analogous* to projection. We can only assert that there will *always*, in *every state* of the Figure, or of the *Net*, be *some point* P, possessing the above-mentioned property: or rather that there will be a *circle* of such points *in space*, having for its *axis* the *line* to which the three segments belong.

[91.] To fix a little more definitely the conceptions, let A, B, C, D be supposed, for the moment, to be the *corners* of a *regular pyramid*, with E for its *mean point*, or *centre of gravity.* With this *arrangement* of the five *given* points P_0, *six* of the *derived* points P_1, namely A', B', C', A_2, B_2, C_2, *bisect* the *six edges* BC, CA, AB, DA, DB, DC, of the given pyramid; and the *four* other points P_1, namely A_1, B_1, C_1, D_1, are the *mean points* of the *four faces*, opposite to A, B, C, D. *Six* of the ten points $P_{2,1}$, namely A'', B'', C'', A_2', B_2', C_2', are now *infinitely distant*; and the *line* $A''B''C''A^{IV}B^{IV}C^{IV}$ to which three of the lately mentioned *segments* belong, becomes the *line at infinity* in the plane ABC: which might seem, at first

* The determination of these double points gives rise naturally to some new theorems, which cannot conveniently be stated here.

sight, to render difficult, with respect at least to *them*, the verification of a recent theorem [89.]. That theorem is, however, verified in a very simple manner, by observing that, with the arrangement here conceived, *the three angles* $A''D_1A^{IV}$, $B''D_1B^{IV}$, $C''D_1C^{IV}$, which those *infinite* and *infinitely distant segments* may be imagined to *subtend* at the point D_1, *are all right angles*; D_1A'', for example, being *parallel* to the *side* BC of the *triangle* ABC, which is now an *equilateral* one; while D_1A^{IV} is *perpendicular* to the same side, because it is drawn from the *mean point* D_1, and passes through the *opposite corner*, A. As another verification of the theorem [89.], it will be found that, with the arrangement here supposed, the *segments* $A'A^{IX}$, $C_0B_1^{IV}$, $C_1^{V}B^{VI}$, *of the seventh trace* [43.], *subtend right angles at the given point* B.

[92.] The *involution of the three segments* [82.] is only *one* of the consequences of the *three harmonic equations* [81.], or of what we have called in [85.] the *Involution of the two Triads*, *abc* and $a'b'c'$. We can therefore *infer more*, respecting the *geometrical relations* of the *six points*, even in the *general state* of the whole *Figure*, or NET, than merely that those three segments subtend *right angles*, as above, *at every point of one real circle*, which has its *centre on the common line*, and its *plane perpendicular thereto.* The *order of succession* of the six points being supposed to be the following, $ac'ba'cb'$, from which it can only differ, if at all, by changes not important to the argument, let P be, as in [90.], a point such that the angles aPa', bPb', cPc' are *right.* Then, because the *three pencils*,

$$P\,.\,ac'bc, \quad P\,.\,c'ba'b', \quad \text{and} \quad P\,.\,ba'ca,$$

are all *harmonic* pencils by [81.], it follows that (with the supposed *order* of the points) the lines Pc' and Pc are respectively the *internal* and *external bisectors* of the *angle* aPb; Pb and Pb', of the angle $c'Pa'$; and Pa', Pa, of bPc: the line Pc bisecting also the angle $a'Pb'$ internally. Hence it is easy to infer the following *continued equation between angles* (which is supposed to be new):

$$aPc' = c'Pb = bPa' = a'Pc = cPb' = \frac{\pi}{6};$$

and therefore we may enunciate this *Theorem*:—"*When six collinear points form a system of two triads in involution, their five successive intervals subtend angles each equal to the third part of a right angle, at every point of a certain circle, of which the axis is their common line.*"

For example, with the particular arrangement [91.] of the five initial points A … E, it is found that the five successive portions, C_0A^{IX}, $A^{IX}C_1^{V}$, $C_1^{V}B_1^{IV}$, $B_1^{IV}A'$, $A'B^{VI}$, of the seventh trace, subtend each an angle of *thirty degrees*, at the given point B; and the six lines D_1A'', D_1C^{IV}, D_1B'', D_1A^{IV}, D_1C'', D_1B^{IV}, if suitably distinguished from their own opposites, succeed each other at angular intervals, of the same common amount.

[93.] In general, if *three equally inclined diameters* of a circle, forming a regular and *six-rayed star*, be taken as a *given triad of lines* [85.], the *triad in involution* therewith is represented by that *other star* of the same kind, of which the diameters *bisect the angles* between those of the former star: so that if we consider any *six successive rays* of the *compound* or *twelve-rayed star*, which results from the combination of these *two*, their *successive angles* are evidently each equal to thirty degrees. But now we see further, that if a *star* of this last kind be *cut in six points* by an *arbitrary transversal* in its plane; and if these six points of section be in any manner put into perspective, by any *new* pencil and transversal: the *six new points*, thus obtained, as forming still *two triads in involution*, must admit of having their *five successive intervals seen*, from *every point* of some *new circle*, under *angles still equal each* to the same *third part of a right angle.*

[94.] We have not yet considered the arrangement of the six points on either the *fifth* or the *sixth* typical *trace* [43.]; but it is easy to do this as follows. Let $a\,b\,c\,\alpha\,\beta\,\gamma$ denote, as new temporary symbols, either the six points of the fifth trace (comp. [58.]),

$$\text{I.}\quad a = (1\,0\,0),\quad b = (1\,\bar{1}\,1),\quad c = (1\,1\,\bar{1}),\quad \alpha = (0\,1\,\bar{1}),\quad \beta = (2\,1\,\bar{1}),\quad \gamma = (2\,\bar{1}\,1);$$

or these six other points, belonging to the sixth trace,

$$\text{II.}\quad a = (1\,1\,1),\quad b = (1\,0\,2),\quad c = (1\,2\,0),\quad \alpha = (0\,1\,\bar{1}),\quad \beta = (2\,3\,1),\quad \gamma = (2\,1\,3);$$

we shall then have, in each case, the three harmonic equations,

$$(b\,a\,c\,\alpha) = (c\,\beta\,a\,\alpha) = (a\,\gamma\,b\,\alpha) = -1.$$

In *each* case, therefore, we may consider ourselves as first deriving from three points a fourth, as the harmonic conjugate of the first with respect to the other two; and then deriving a fifth point, and a sixth, as the harmonic conjugates of that fourth point, with respect, on the one hand, to the third and first points; and on the other hand, to the first and second points of the system.

[95.] Having regard merely to this *common law*, we may enunciate (comp. [80.] [81.]) this theorem:—

"*The sixty lines, in the ten planes of first construction, represented by the fourth and fifth typical traces of the planes on the plane* ABC*, although not all syntypical, are all homographically divided.*"

And this *common mode* of their *division* is such, that if the fourth point be thrown off to infinity, the first point bisects the interval between the second and third; the fifth point bisects the interval between third and first; and the sixth point bisects the interval between first and second: so that, on the whole, we have a *finite line*, *bc*, *quadrisected* in the points γ, a, β, and cut at infinity in α; whereas if, on either the *fourth* or the *seventh* trace, *one* of the six points, but only one, had been thus made *infinitely distant*, the *five others* would have presented the figure of a *finite right line, bisected* and *trisected.* With the equations [94.], if a, instead of α, be projected to infinity, it is then the line $\beta\gamma$ which is quadrisected, namely, in the points c, α, b. In general, with these last equations, the *first set* of three points, $a\,b\,c$, can be *derived* from the *second set*, $\alpha\,\beta\,\gamma$, by the *same rule* [94.], as that by which the second set has been derived from the first: so that there is a sense in which *these two sets* may be said to be *reciprocal triads*, although they are *not triads in involution*, according to the definition [85.].

[96.] It may be added that, on either the *fifth* or the *sixth* trace, the two points which we have called *first* and *fourth*, are the *double points* of a new *involution*, determined by the *two pairs*, *second* and *third*, *fifth* and *sixth*; or, with the recent notations [94.], that $a\,\alpha$ are the *foci* of the involution bc, $\beta\gamma$; because the three last harmonic equations conduct to this fourth equation,

$$(\beta\, a\gamma\alpha) = -1.$$

[97.] And, as regards the *homography* of the divisions on the same two traces, if we denote, for the sake of distinction, the six points on the sixth trace by $a' \ldots \gamma'$, then (because $\alpha' = \alpha$) the *five lines* aa', bb', cc', $\beta\beta'$ $\gamma\gamma'$, or (comp. [58.]) the five lines

$$\mathrm{AD}_1, \quad \mathrm{B}_0\mathrm{B}^{\mathrm{V}}, \quad \mathrm{C}_0\mathrm{C}_1^{\mathrm{V}}, \quad \mathrm{B}_1^{\mathrm{VII}}\mathrm{B}_1^{\mathrm{VIII}}, \quad \mathrm{C}^{\mathrm{VII}}\mathrm{C}^{\mathrm{VIII}},$$

ought to *concur* in some *one point*: which accordingly it is easy to see that they do, namely in the point A′; in fact, with the recent signification of $a, \ldots$ and $a', \ldots$, we have the symbolic equations,

$$(a') - (a) = (b') - (b) = (c') - (c) = (0\,1\,1) = (\mathrm{A}');$$

and

$$(\beta') - (\beta) = (\gamma') - (\gamma) = (0\,2\,2) = 2(\mathrm{A}').$$

[98.] The *two sets of six points*, on these two traces, with one point common, are thus the points in which a certain *six-rayed pencil*, with A′ for vertex, is *cut* by the two traces as transversals; the *symbols* of the six *rays* being the following:

$$\mathrm{A}'\mathrm{AD}_1 = [0\,1\,\bar{1}]; \quad \mathrm{A}'\mathrm{B}_0\mathrm{B}^{\mathrm{V}} = [\bar{2}\,\bar{1}\,1]; \quad \mathrm{A}'\mathrm{C}_0\mathrm{C}_1^{\mathrm{V}} = [\bar{2}\,1\,\bar{1}];$$

$$\mathrm{A}'\mathrm{A}'' = [1\,0\,0]; \quad \mathrm{A}'\mathrm{B}_1^{\mathrm{VII}}\mathrm{B}_1^{\mathrm{VIII}} = [1\,\bar{1}\,1]; \quad \mathrm{A}'\mathrm{C}^{\mathrm{VII}}\mathrm{C}^{\mathrm{VIII}} = [1\,1\,\bar{1}].$$

And from a mere inspection of these symbols, we can infer (comp. (33)) that the *first* and *fourth rays* are the *common harmonic conjugates* of the *two pairs*, *second* and *third*, *fifth* and *sixth*; or that they are the *double rays* of the *involution*, which those two *pairs of rays* determine: the theorem [96.] being thus, in a new way, confirmed.

[99.] We have now discussed the arrangements of the *points* on those *nine* typical *lines* Λ_3, whereof each passes through not less than *four*, nor more than *six*, of the 52 points in the plane A B C; but we have still *three other typical lines* to consider, namely the lines Λ_1 and Λ_2, of which each passes through *at least seven points*. Taking first, for this purpose, the typical line $\Lambda_{2,1}$, namely AA′, which contains *only seven* points, whereof the ternary symbols have been assigned in [55.], and the literal symbols there given may be retained, we shall, for the moment, reserve the consideration of the two points $\mathrm{P}_{2,3}$; but shall introduce a new and auxiliary point $\mathrm{P}_{3,1}$ on the same line, which may be thus denoted:

$$\mathrm{A}^{\mathrm{X}} = (1\,2\,2) = \mathrm{AA}' \cdot \mathrm{BC}''' \cdot \mathrm{CB}''';$$

and which may be said to *represent* or *typify* a *first group of third construction*, containing *fifteen points*, *one* on each of the *fifteen lines* $\Lambda_{2,1}$; although, in the present Paper, we can only *allude* to *such* new points P_3, and cannot *here* attempt to *enumerate*, or even to *classify* them.

[100.] We have thus again *six* points, at this stage, to consider, namely the points A, A′, D_1, A‴, A_0, A^X; and their symbols easily show that they are connected by the *three* following harmonic equations,

$$(\mathrm{A\,A'\,D_1\,A'''}) = (\mathrm{A\,D_1\,A'\,A_0}) = (\mathrm{A'\,A\,D_1\,A^X}) = -1;$$

from which it follows, by [85.], that the *two triads of points*,

$$\mathrm{AA'D_1} \quad \text{and} \quad \mathrm{A^X A''' A_0},$$

are triads in involution: with, of course, all the properties which have been proved, in recent paragraphs of this Paper, to belong generally to *any two* such triads. As a verification, it may be mentioned that, with the particular arrangement [91.] of the five initial points A ... E, if we determine two new points P, P′, of *third* construction, by the formulæ,

$$\mathrm{P} = (2\,1\,4) = \mathrm{BC'''} \cdot \mathrm{CA'''}, \quad \mathrm{P'} = (2\,4\,1) = \mathrm{CB'''} \cdot \mathrm{BA'''},$$

it can be proved that each of the five successive intervals (comp. [92.]) between the six points,

$$\mathrm{A},\ \mathrm{A'''},\ \mathrm{D_1},\ \mathrm{A^X},\ \mathrm{A'},\ \mathrm{A_0},$$

subtends the third part of a right angle at each of these two new auxiliary points, P and P′. But with *other* initial configurations, the *coordinates* of these two *new vertices* would be different, because they are connected with *angles*, which are not generally *projective* [90.]; althrough, as has been already remarked, there would always be *some* new points P, or rather a *circle* of such, possessing the property in question.

[101.] We may however enunciate generally, and without reference to any such particular *arrangement* of the five initial points, this *Theorem*:—

"*On any one of the fifteen lines* $\Lambda_{2,1}$*, of second construction, and first group, the given point* P_0*, and the two derived points of first construction* P_1*, compose a triad, the triad in involution to which* [85.] *consists of the point* $P_{3,1}$*, of third construction and first group, and of the two points* $P_{2,2}$*, of second construction and second group, upon that line*;" with *seven involutions of segments* (comp. [84.]) *included* under this general relation.

For example, on the line AA′, the *three segments* AA^X, A′A‴, D_1A_0 form always an *involution* of the *ordinary* kind, with its *double points imaginary*; the *three other sets* of segments, AA^X, $A'A_0$, D_1A'''; A′A‴, AA_0, D_1A^X; and D_1A_0, AA‴, $A'A^X$, form *each* an involution, with *real double points*; the points A, A^X are the *real foci* of a *fifth* involution, determined by the *two pairs* of segments $A'D_1$, and $A'''A_0$; the points A′, A‴ are, in like manner, the real double points of that *sixth* involution, which the *two other pairs*, A, D_1, and A_0, A^X, determine: and finally, D_1 and A_0 are such points, for the *seventh* involution, determined by AA′, $A'''A^X$.

[102.] Introducing now the consideration of the two lately *reserved points* $P_{2,3}$ [99.] of *second construction* and *third group* [45.], upon the typical line $\Lambda_{2,1}$, we may derive them from the point P_0, the two points P_1, and the two points $P_{2,2}$ upon that line AA', by the two following harmonic equations:

$$(A\,A'''\,A'\,A^{IV}) = (A\,A_0\,D_1\,A_1^{IV}) = -1;$$

or by these two others,

$$(A\,A'\,A_0\,A^{IV}) = (A\,D_1\,A'''\,A^{IV}) = -1,$$

which may indeed by inferred from the two former, with the help of the relations between the six points previously considered: for, in general, if abc, $a'b'c'$ be collinear triads in involution, and if d and d' be the harmonic conjugates of b' and c', with respect to the two pairs, ab, ac, they are also the harmonic conjugates of b and c, with respect to the two *other* pairs, ac', ab'; or in symbols,

$$(a\,b\,c'\,d) = (a\,c\,b'\,d') = -1, \quad \text{if} \quad (a\,b'\,b\,d) = (a\,c'\,c\,d') = -1,$$

when the three harmonic equations [81.] exist. We have also, generally, under these conditions, the equation

$$(a\,d\,a'\,d') = -1;$$

for example, on the line AA', we have

$$(A\,A^{IV}\,A^{X}\,A_1^{IV}) = -1.$$

[103.] It is scarcely worth while to remark that the 15 lines $\Lambda_{2,1}$ of the net, as being all *syntypical*, are all *homographically divided*; although it may just be noticed, as a verification, that the six lines,

$$BC, \quad B'C', \quad B'''C''', \quad B_0C_0, \quad B^{IV}C^{IV}, \quad B_1^{IV}C_1^{IV},$$

which connect corresponding points on the two other lines of the same group in the given plane, namely $BB'D_1$ and $CC'D_1$, *concur* in one point A''. But it may not be without interest to observe, that A^X is the *common harmonic conjugate* of A, with respect to *each* of the three pairs, $A'D_1$, $A'''A_0$, $A^{IV}A_1^{IV}$; which *three pairs*,* or segments, form thus an *involution*, with A and A^X for its *double points*. We have therefore this *Theorem*:—

"*On each of the fifteen lines* $\Lambda_{2,1}$*, the three pairs of derived points, of first and second constructions*, namely the *pair* P_1*, the pair* $P_{2,2}$*, and the pair* $P_{2,3}$*, compose an involution, one double point of which is the given point* P_0*; the other double point being the point* $P_{3,1}$*, of third construction and first group, upon the line.*"

* That the *two first* of these three pairs belong to an involution, with those two double points, was seen in [101.].

[104.] We have thus discussed the arrangements of the points P_0, P_1, P_2, on each of the *ten* typical lines which connect not *fewer* than *four*, and not *more* than *seven* of them; but there are still *two other* typical lines to be considered, belonging to the groups Λ_1 and $\Lambda_{2,2}$, whereof one, as BC, passes through *eight* points [54.]; and the other, as B′C′, has *ten* points upon it [56.]. Beginning with the first, we easily find that the two sets of points A′BC and $A''A_1^{V}A^{V}$, are *triads in involution* [85.]; the latter set being thus deducible from the former: while the two other points upon the line may be determined by the condition that they satisfy this *other involution of two triads*, A″BC, $A'A_1^{VI}A^{VI}$. With the *initial arrangement* [91.], the line $A^{VI}A_1^{VI}$ is *trisected* in B and C, and its *middle part* BC is likewise *trisected* in A^{V} and A_1^{V}; while *each* line is *bisected* in A′, and *cut at infinity* in A″. And in general we may enunciate these *two Theorems*:—

I. "*On every line of first construction, the point* P_1 *and the two points* P_0 *form a triad, the triad in involution with which consists of the point* $P_{2,1}$*, and the two points* $P_{2,4}$."

II. "*On every such line* Λ_1*, the triad formed by the point* $P_{2,1}$*, and the two points* P_0*, is in involution with a triad which consists of the point* P_1 *and the two points* $P_{2,5}$."

[105.] Besides these *two involutions of triads*, we have *two distinct involutions* of the *ordinary* kind, into *each* of which *all the eight points enter*; two being *double points* in each. For we have these *two other Theorems*, deducible, indeed from the two former, but perhaps deserving to be separately stated:—

III. "*On every line of first construction, the two given points are foci of an involution of six points, in which the points* P_1, $P_{2,1}$*, are one pair of conjugates, while the two other pairs are of the common form,* $P_{2,4}$, $P_{2,5}$." For example, A^{V}, A^{VI} are such a pair, on the line BC.

IV. "*On every such line* Λ_1*, the points* P_1, $P_{2,1}$*, are the double points of a second involution of six points, obtained by pairing the two points of each of the three other groups.*"

[106.] Finally, as regards the *remaining typical line* B′C′, which connects *two points* P_1, and passes through *eight points* P_2, if we reserve for a moment the consideration of the *last pair*, $P_{2,8}$, or A^{IX} and A_1^{IX}, we have a *system of eight points upon that line, homographic with the recent system of eight points on the line* BC; being indeed the *intersections* of the line B′C′ with the *eight-rayed pencil*, $A\,.\,A'BCA''A_1^{V}A^{V}A_1^{VI}A^{VI}$, when taken in the order $A'''C'B'A''A_1^{VIII}A^{VIII}A_1^{VII}A^{VII}$. No description of the arrangement of these latter *points* is therefore at this stage required: but as regards the *pencil*, it may be remarked that, by [104.], the 1st, 2nd, and 3rd *rays* form a *triad of lines, in involution* [85.] *with the triad* formed by the 4th, 5th and 6th; and that the *triad* of the 2nd, 3rd and 4th rays is, in the same new sense, in *involution* with the *triad* of the 7th, 8th, and 1st: from which *double involution of triads*, the *five last rays* may be *derived*, if the *three first* are *given.* We have also by [105.] a *double involution of the rays*, considered as *paired* with *each other*, or with *themselves*: thus the second and third rays are the *double rays* of an involution (or the *usual* kind), in which the first is conjugate to the fourth, the fifth to the seventh, and the sixth to the eighth; while the first and fourth rays are the double rays of *another* involution, in which the second and third, the fifth and sixth, and the seventh and eighth are conjugate.

[107.] It only remains to assign the arrangement of the *two last points of second construction*, $P_{2,8}$, with respect to the *other points* P_1, P_2, on a line $\Lambda_{2,2}$, or to some *three* of them; or to show how A^{IX} and A_1^{IX} can be *derived*,* for example, from B', C', and A'': which derivation may easily be effected, on the plan already described for the fifth and sixth typical traces. In fact, if we denote the six points $A''\,C'\,B'\,A'''\,A_1^{IX}\,A^{IX}$ by $a\,b\,c\,\alpha\,\beta\,\gamma$, we have the three harmonic equations of [94.]; and if, by one of the modes of *perspective*, or *projection*, mentioned in [95.], which answers to the initial arrangement [91.], we throw off the first point A'' to *infinity*, the finite line $A^{IX}A_1^{IX}$ is then *quadrisected*: being *itself bisected* at A''', while C' and B' *bisect its halves*. In general, we shall have again the equations [94.], if we otherwise represent the six lately mentioned points on $B'C'$ by $\alpha\,\beta\,\gamma\,a\,b\,c$; and thus it is seen that *those six points* are *always homographic, in every state* of the figure, or *net*, with the six points $A''\,B_1^{VII}\,C^{VII}\,A\,B_0\,C_0$ on the *fifth trace* AA'', and with the six points $A''\,B_1^{VIII}\,C^{VIII}\,D_1\,B^{V}\,C_1^{V}$ on the *sixth trace* D_1A''; in fact they are, if taken in a suitable order, the points in which the *six-rayed pencil* [98.], with A' for vertex, is cut by the line $B'C'$.

[108.] We have thus shown for each of the *twelve typical lines* [74.], in the plane ABC, how *all the points but three*, upon that line, may be derived *from those three* by a *system of harmonic equations*, not *necessarily* employing any point P_3, or other *foreign*† or merely *auxiliary point*: although it appeared that something was gained, in respect to elegance and clearness, by introducing, on the line AA', such a point A^{x} [99.]; or by considering generally, on any one of the fifteen lines $\Lambda_{2,1}$, a point $P_{3,1}$ of *third construction*, belonging to what may perhaps deserve to be regarded as a *first group* [103.] of the points P_3, in any future *extension* [1.] of the results of the present Paper.

Part V.—*Applications to the Net, continued: Distribution of the Given or Derived Points, in a Plane of Second Construction, and of First or Second Group.*

[109.] It will be necessary to be much more concise, in our remarks on the distribution of the *net-points* in *planes* of *second construction*; but a few general remarks may here be offered, from which it will appear that each plane $\Pi_{2,1}$ contains *forty-seven* of the 305 points P_0, P_1, P_2; and that each plane $\Pi_{2,2}$ contains *forty-three* of those points; with many cases of *collineation* for each.

[110.] We saw in [33.], that each plane $\Pi_{2,1}$ contains two lines $\Lambda_{2,1}$, which intersect in a point P_0, and may be regarded as the diagonals of a quadrilateral, of which the four sides are lines $\Lambda_{2,2}$. It contains, therefore, as has been seen, one point P_0, and four points P_1; but it is found to contain also 42 points P_2, arranged in *six groups*, as follows.

* This point A^{IX} may also, by [81.], be determined on the *seventh trace*, or *seventh typical line* [74.], as the *harmonic conjugate* of A', with respect to C_0 and C_1^{V}.

† This *non-requirement* of *foreign points* is the only remarkable thing here: for the *anharmonic function* of *every group* of *four collinear net-points* is necessarily *rational*; and whenever $(a\,b\,c\,d)$ = any positive or negative quotient of *whole numbers*, it is *always possible* to deduce the *fourth point* d from the *three* points a, b, c, by *some system* of *auxiliary points*, derived successively from them through *some system of harmonic equations*.

[111.] There are 2 points $\text{P}_{2,1}$, namely the intersections of opposite sides of the quadrilateral; thus, in what we have called the *second typical plane* [33.], the sides B_1C_1, C_2B_2 intersect in the point A''; and the sides C_1C_2, B_2B_1 in D'_1 (62).

[112.] The plane contains also 8 points $\text{P}_{2,2}$; namely, *two* on each of the *two diagonals*, and *one* on each of the *four sides*; and it contains 4 points $\text{P}_{2,3}$, namely two on each diagonal: but it contains *no* point of either of the two groups $\text{P}_{2,4}$, $\text{P}_{2,5}$, as a comparison of their *types* sufficiently proves, or as may be inferred from the *laws* of their construction [46.] [47.].

[113.] The same plane contains 12 points $\text{P}_{2,6}$; namely two on each side of the quadrilateral; and four others, in which the plane is intersected by four lines $\Lambda_{2,2}$; as the *types* sufficiently prove. But to show, geometrically, *why* there should be *only four such intersections*, conducting thus to new points $\text{P}_{2,6}$ in the plane, let the five inscribed pyramids [28.] be denoted by the symbols $\text{A}' \ldots \text{E}'$; then the six edges of the pyramid A' are found to intersect the present plane $\Pi_{2,1}$ in points already considered, namely in the two points $\text{P}_{2,1}$, of *meetings of opposite sides*, and in those four points $\text{P}_{2,2}$, which are situated *on the diagonals* of the quadrilateral; they give therefore *no new points*. Also, each *side* of the same quadrilateral is an *edge* of one of the *four other pyramids* $\text{B}' \ldots \text{E}'$; but there remains, for each such pyramid, an *opposite edge*: and these are the *four lines, out of the plane*, which *intersect it* in the *four points* $\text{P}_{2,6}$, additional to the *eight points* $\text{P}_{2,6}$, which are ranged, two by two, *upon the sides*. There are thus *twelve points* of the *group* $\text{P}_{2,6}$, in any one plane $\Pi_{2,1}$; and we have now exhausted the intersections of that plane with the lines $\Lambda_{2,2}$; and also, as it will be found, with the lines $\Lambda_{2,1}$, and Λ_1.

[114.] But there remain *eight* points $\text{P}_{2,7}$, and *eight* points $\text{P}_{2,8}$, in the plane now considered; namely *two* of *each group*, on each of the *four sides* of the quadrilateral. There are, therefore, 16 such points; which, with the 12 points $\text{P}_{2,6}$; the 4 points $\text{P}_{2,3}$; the 8 points $\text{P}_{2,2}$; the 2 points $\text{P}_{2,1}$; the 4 points P_1; and the one point P_0, make up (as has been said in [109.]) a system of 47 points, *given or derived*, in any one of the fifteen planes $\Pi_{2,1}$.

It may be remarked that with the initial arrangement [91.] of the five given points, the four points $\text{B}'\ \text{C}'\ \text{B}_2\ \text{C}_2$, in a new plane $\Pi_{2,1}$, are corners of a *square*, which has the point E for its *centre*; and that thus the Figure, of the 47 points in such a plane, may be thrown into a clear and elegant perspective.

[115.] As regards the distribution in a plane $\Pi_{2,2}$, such as the *Third Typical Plane* [34.], it may here be sufficient to observe, that besides containing *three lines* $\Lambda_{2,2}$, namely the *sides of a triangular face* [34.] of one of the *five inscribed pyramids* [28.], and *three points* P_1, which are the *corners* of that *triangle*, and serve to *determine the plane* [1.], it contains also *forty points* P_2, which are arranged in *groups*, as follows. *Each* of the *four first groups*, of *second construction*, $\text{P}_{2,1}, \ldots \text{P}_{2,4}$, gives *three points* to the plane; the *fifth group*, $\text{P}_{2,5}$, furnishes only *one point*; and the *sixth*, *seventh* and *eighth* groups, $\text{P}_{2,6}, \ldots \text{P}_{2,8}$, supply *six*, *twelve*, and *nine* points, respectively. Of these 40 points P_2, *twenty-four* are ranged, eight by eight, *on the three sides* of the triangle, as was to be expected from [56.]; and the existence of *at least* 27 *points*, P_1, P_2, in a plane $\Pi_{2,2}$, might thus have been at once foreseen. But we have also to consider the *traces*, on that plane, of the 52 *lines*, Λ_1, Λ_2, which are not contained therein.

Of these lines, it is found that 36 *intersect the sides* of the triangle, and give therefore *no new points*. But the *sixteen other lines* intersect the *plane*, in so many *new* and *distinct points*; and thus the *total number* [109.], of *forty-three derived points*, P_1, P_2, in a plane $\Pi_{2,2}$, which contains *no given point* P_0, is made up.

[116.] Without attempting here to enumerate the cases of *collineation*, in either of the two typical planes Π_2, we may just remark, that while the traces of four of the planes Π_1 on the typical plane $\Pi_{2,1}$ are the four sides, and the traces of four others are the diagonals, of the quadrilateral already mentioned, the trace of a ninth plane Π_1, namely ABC, on that plane $\Pi_{2,1}$ has been already considered, as the trace AA'' of the latter on the former; but that the trace of the *tenth plane* Π_1, namely ADE, or $[0\,1\,\overline{1}\,0\,0]$, on $\text{AB}_1\text{C}_2\text{C}_1\text{B}_2$, or on $[0\,1\,1\,\overline{1}\,\overline{1}]$, is a *new line*, AD'_1; which passes thus through one point P_0 and one point $\text{P}_{2,1}$, and also through two points $\text{P}_{2,2}$, namely $(0\,1\,1\,2\,0)$ and $(0\,1\,1\,0\,2)$, and through two points $\text{P}_{2,6}$, namely $(2\,0\,0\,1\,\overline{1})$ and $(2\,0\,0\,\overline{1}\,1)$: being, however, *syntypical* with the formerly considered trace AA'', and therefore leading to no new harmonic or anharmonic relations.

[117.] As a specimen of a case of collineation which conducts to such *new relations*, let us take the four following points P_2, in the second typical plane,

$$a = (0\,1\,1\,2\,0), \quad b = (0\,0\,2\,1\,1), \quad c = (0\,2\,0\,3\,\overline{1}), \quad d = (0\,\overline{1}\,3\,0\,2),$$

whereof the two first are points $\text{P}_{2,2}$, and the two last are points $\text{P}_{2,8}$; and of which the symbols satisfy the equations,

$$(c) = 2(a) - (b), \quad (d) = -(a) + 2(b); \quad \text{whence} \quad (a\,d\,b\,c) = 4.$$

These four points, therefore, with which it is found that *no other* given or derived point of the system P_0, P_1, P_2 is *collinear*, do *not* form a *harmonic group*; and consequently we *cannot construct the fourth point*, d, when the *three other* points, a, b, c, are *given*, by means of *harmonic relations alone* (comp. [108.]), unless we introduce some *auxiliary point*, or points, $e, \ldots$, which shall be at lowest of the *third construction.* But if we write

$$e = (1\,2\,0\,2\,0) \equiv (0\,1\,\overline{1}\,1\,\overline{1}), \quad f = (\overline{1}\,0\,2\,2\,0) \equiv (0\,1\,3\,3\,1),$$

so that e is a point $\text{P}_{3,1}$ [99.], while f may be said to be a point $\text{P}_{3,2}$, we find that these two *new* or *auxiliary* points, e, f, are the *double points* of the *involution*, determined by the *two pairs*, ab, cd; because we have the two harmonic equations,

$$(a\,e\,b\,f) = (c\,e\,d\,f) = -1.$$

And because we have also,

$$(c\,a\,b\,e) = (a\,b\,d\,e) = -1,$$

we need only employ the *one* auxiliary point e, considered as the harmonic conjugate of a, with respect to b and c; and then determine the fourth point d, as the harmonic conjugate of a, with respect to b and e. It may be added that abe and dcf are *triads in involution* [85.]; so that if e be projected to infinity, the finite line cd is *trisected* at a and b.

PART VI.—*On some other Relations of Complanarity, Collinearity, Concurrence, or Homology, for Geometrical Nets in Space.*

[118.] Although we have not proposed, in the present Paper, to *enumerate*, or even to *classify*, any points, lines, or planes, beyond what we have called the *Second Construction* [1.], yet *some* such points, lines, and planes have offered themselves naturally to our consideration: and we intend, in this *Sixth Part*, to consider a few others, chiefly in connexion with relations of *homology*, of triangles or pyramids which have been already mentioned.

[119.] It was remarked in [29.], that the thirty lines $\Lambda_{2,2}$ are the sides of *ten triangles* T_2, of *second construction*, which are certain *inscribed homologues* of ten *other* triangles T_1, of *first* construction [26.]; the *ten* corresponding *centres* of homology being the ten points P_1. For example, the triangle $\mathrm{A'B'C'}$ is inscribed in ABC, and is *homologous* thereto, the point D_1 being their *centre* of homology; because we have the three relations of *intersection*,

$$\mathrm{A}' = \mathrm{D_1A} \cdot \mathrm{BC}, \ \&c.;$$

or because, A' being a point of BC, &c., the *three joining lines* AA', &c., *concur* in the point D_1.

[120.] Proceeding to determine the *axis* of this homology, or the right line which is the locus of the points of intersection of corresponding sides, we easily see that it is the line $\mathrm{A''B''C''}$; because we had $\mathrm{A''} = \mathrm{BC} \cdot \mathrm{B'C'}$, &c. And because an analogous result must take place in *each* of the *ten planes* Π_1, we see that *the ten points* $\mathrm{P}_{2,1}$ *are ranged, three by three, on ten lines* $\Lambda_{3,1}$*, in the ten planes* Π_1; namely on the *axes of homology* of the *ten pairs of triangles* T_1, T_2, in those ten planes: which axes are the lines,

$$\mathrm{D'_1A'_1A'_2}, \ \&c.; \quad \mathrm{C'_1B'_1A''}, \ \&c.; \quad \mathrm{C'_2B'_2A''}, \ \&c.; \quad \text{and} \quad \mathrm{A''B''C''};$$

each point $\mathrm{P}_{2,1}$ being thus *common* to *three* of them, because it is common to those *three planes* Π_1, which contain the line Λ_1 whereupon it is situated. Each point $\mathrm{P}_{2,1}$ is also the *common intersection* of this last line with *three lines* $\Lambda_{2,2}$; we have for example, the *formulæ of concurrence*,

$$\mathrm{A''} = \mathrm{BC} \cdot \mathrm{B'C'} \cdot \mathrm{B_1C_1} \cdot \mathrm{B_2C_2}.$$

[121.] The line $\mathrm{A''B''C''}$ was seen to be the *common trace* of *two planes* $\Pi_{2,2}$, namely of $\mathrm{A_1B_1C_1}$ and $\mathrm{A_2B_2C_2}$, on the plane Π_1, namely ABC, in which it is situated; and a similar result must evidently hold good for *each* of the *ten lines* $\Lambda_{3,1}$. But we may add that the *three triangles* ABC, $\mathrm{A_1B_1C_1}$, $\mathrm{A_2B_2C_2}$, in the plane of *each* of which the line $\mathrm{A''B''C''}$ is contained, are *homologous, two by two*, and have this line for the *common axis of homology* of each of their *three pairs*; having however *three distinct centres* of homology, namely D'_1 for second and third, D for third and first, and E for first and second: with (as we need not again repeat) analogous results for the *other* lines $\Lambda_{3,1}$, of which *group* we here take the line $\mathrm{A''B''C''}$ as *typical.* It may be remarked that the *four centres*, recently determined, are *collinear*, and compose an *harmonic group*; and that the *inscribed* triangle $\mathrm{A'B'C'}$ is also *homologous* with *each* of the two triangles $\mathrm{A_1B_1C_1}$, $\mathrm{A_2B_2C_2}$, although not *complanar* with *either*; the line $\mathrm{A''B''C''}$ being *still* the *common axis* of homology; while the *two centres*, of these last two homologies, are the two given points, D and E.

[122.] The *six points* $P_{2,2}$, in the plane ABC, have been seen to range themselves, according to their *two ternary types* [41.], into *two sets of three*, which are the *corners* of *two new triangles*; *one* of these, namely $A'''B'''C'''$, being an *inscribed homologue* of $A'B'C'$; while the *other*, namely $A_0B_0C_0$, is an *exscribed homologue* of ABC; and these two triangles are also homologous to *each other*: the *line* $A''B''C''$ being still the *common axis*, and the *point* D_1 being the *common centre* of homology. And the same thing holds good for any one of these four triangles, $A_0B_0C_0$, ABC, $A'B'C'$, $A'''B'''C'''$, in the plane Π_1 here considered, as compared with the triangle $A_1^{IV}B_1^{IV}C_1^{IV}$, whereof the corners are those three points $P_{2,3}$, which are *not* ranged on the line $A''B''C''$, as the three *other* points $P_{2,3}$, namely A^{IV}, B^{IV}, C^{IV}, have been seen to be.

[123.] It was remarked in [28.], that each of the *five pyramids* R_2 is not only *inscribed* in the corresponding pyramid R_1 [26.], but it is also *homologous* therewith; the *centre* of their homology being a point P_0: thus the point E is such a centre, for the two pyramids ABCD and $A_1B_1C_1D_1$, or for those which we have lettered as E and E′ [26.] [113.]. The *planes* BCD, $B_1C_1D_1$, of two corresponding *faces*, intersect in the *line* $C_2'B_2'A''$; the planes CAD, $C_1A_1D_1$, in $A_2'C_2'B''$; the planes ABD, $A_1B_1D_1$, in $B_2'A_2'C''$; and the planes ABC, $A_1B_1C_1$, in $A''B''C''$. Hence it is easy to infer that *these six points* $P_{2,1}$, namely

$$A'',\ B'',\ C'',\ A_2',\ B_2',\ C_2',$$

are all situated *in one plane*, which is the *plane of homology* of the *two pyramids* E and E′, and which we shall denote by [E]; its *quinary symbol* being

$$[E] = [1\,1\,1\,1\,\overline{4}],$$

which may also serve as a *type* of the *group* [A] ... [E]. And in fact, the quinary symbols of the six points all satisfy the *equation* (comp. [19.],

$$x + y + z + w = 4v.$$

[124.] It may be noted that the *two planes* of homology, [D] and [E], have the *line* $A''B''C''$ for their *common trace* on the plane ABC; and that the traces of the *three other planes* of the same group, [A], [B], [C], which have

$$[\overline{4}\,1\,1], \quad [1\,\overline{4}\,1], \quad [1\,1\,\overline{4}],$$

for their *ternary symbols*, pass respectively through the points A^X, B^X, C^X, (comp. [99.]), and coincide with the lines $B_1^{IV}C_1^{IV}$, &c., or with the *sides* of the last mentioned *triangle* [122.]. And it follows from [123.], that *the ten points* $P_{2,1}$ are ranged *six by six*, and that the *ten lines* $\Lambda_{3,1}$ are ranged *four by four*, in *five planes* $\Pi_{3,1}$; namely in the five planes [A] ... [E] of *homology of pyramids*. But *these last laws* of arrangement, of points and lines, must be considered as included in results which have been comparatively long known, respecting *transversal* lines and planes in space.*

* Compare the second note to [1.].

[125.] Instead of *inscribing* a pyramid E' in the pyramid E, we may propose to *exscribe* to the latter a *new* pyramid $\mathrm{A}^{\backprime}\mathrm{B}^{\backprime}\mathrm{C}^{\backprime}\mathrm{D}^{\backprime}$, or $\mathrm{E}^{\backprime}$, which shall be *homologous* with it, the given point E being still the *centre* of homology. In other words, the *four new planes* $\mathrm{B}^{\backprime}\mathrm{C}^{\backprime}\mathrm{D}^{\backprime}, \ldots, \mathrm{A}^{\backprime}\mathrm{B}^{\backprime}\mathrm{C}^{\backprime}$, or $\mathrm{E_A}$, $\mathrm{E_B}$, $\mathrm{E_C}$, $\mathrm{E_D}$, are to pass *through the four given points* A, B, C, D; and the *four new lines* $\mathrm{AA}^{\backprime}$, $\mathrm{BB}^{\backprime}$, $\mathrm{CC}^{\backprime}$, $\mathrm{DD}^{\backprime}$ are to *concur*, in the *fifth given point* E. The solution of this problem is found to be expressed by the following quinary symbols for the four sought planes:

$$[\mathrm{E_A}] = [0\,1\,1\,1\,\overline{3}], \quad \ldots \quad [\mathrm{E_D}] = [1\,1\,1\,0\,\overline{3}].$$

In fact, the pyramid $\mathrm{E}^{\backprime}$, with these four planes for *faces* is evidently *exscribed* to the pyramid ABCD, or E; and because its *corners* may be represented by these other quinary symbols,

$$\mathrm{A}^{\backprime} = (3\,0\,0\,0\,1), \quad \ldots \quad \mathrm{D}^{\backprime} = (0\,0\,0\,3\,1),$$

the condition of *concurrence* is satisfied. We may remark that the plane [E] of [123.] is the plane of homology of the two last pyramids E and $\mathrm{E}^{\backprime}$; and that this *exscribed pyramid* $\mathrm{E}^{\backprime}$ is homologous also to the *inscribed* pyramid E', the point E being still the *centre*, and the plane [E] the *plane* of their homology.

[126.] It may be remarked that the *common trace* of the two planes $\mathrm{E_D}$ and $\mathrm{D_E}$, on the plane ABC, is the line $\mathrm{A''B''C''}$; to *construct*, then, the *exscribed pyramid* $\mathrm{E}^{\backprime}$, we may construct the plane $\mathrm{E_D}$ of *one* of its *faces*, by connecting the *point* D with the line $\mathrm{A''B''C''}$; and similarly for the rest. Or if we wish to determine separately the *new point*, or corner, $\mathrm{D}^{\backprime}$, which *corresponds* to the given point D, we may do so, by the *anharmonic equation*,

$$(\mathrm{D\,D_1\,E\,D}^{\backprime}) = 3;$$

for which may be substituted* the system of the *two* following *harmonic* equations:

$$(\mathrm{D\,D_1\,E\,F}) = (\mathrm{D\,D}^{\backprime}\,\mathrm{D_1\,F}) = -1;$$

where F is an auxiliary point, namely D'_1.

Part VII.—*On the Homography and Rationality of Nets in Space; and on a Connexion of such Nets with Surfaces of the Second Order.*

[127.] In general, *all geometric nets in space* are *homographic figures*; *corresponding points, lines*, and *planes*, being those which have the *same* or (*congruent*) *quinary symbols*, in whatever manner we may pass from one to another system of *five initial points*, A ... E; whereof it is still supposed that *no four are complanar. All* points, lines, and planes of any such *Net* are evidently *rational*, in the sense [8] already defined, with respect to the initial system; and conversely it is not difficult to prove that every *rational point, line*, or *plane*, in space, *is a net-point*, *net-line*, or *net-plane*, whatever that initial system of five points may be. It follows that although *no irrational point, line*, or *plane*, can possibly *belong* to the *net*,

* Compare the note to [108.].

with respect to which it *is* thus irrational, yet it can be *indefinitely approached to*, by points, lines, or planes which *do* so belong: a remarkable and interesting theorem, which appears to have been first discovered by *Möbius*;* to whom indeed, as has been already said, the *conception of the net* is due, but whose *analysis* differs essentially from that employed in the present Paper.

[128.] As regards the *passage from one net in space to another*, let the *quinary symbols* of some five given points $\text{P}_1 \ldots \text{P}_5$, whereof no four are in one plane, be with respect to the *given* initial system $\text{A} \ldots \text{E}$ the following:—

$$\text{P}_1 = (x_1 \ldots v_1), \quad \ldots \quad \text{P}_5 = (x_5 \ldots v_5);$$

and let $a' \ldots e'$ and u' be six coefficients, determined so as to satisfy the *quinary equation* [5.],

$$a'(\text{P}_1) + b'(\text{P}_2) + c'(\text{P}_3) + d'(\text{P}_4) + e'(\text{P}_5) = -u'(\text{U}),$$

or the five ordinary equations which it includes, namely,

$$a'x_1 + \cdots + e'x_5 = \cdots = a'v_1 + \cdots + e'v_5 = -u'.$$

Let P' be any sixth point of space, such that

$$(\text{P}') = xa'(\text{P}_1) + yb'(\text{P}_2) + zc'(\text{P}_3) + wd'(\text{P}_4) + ve'(\text{P}_5) + u(\text{U});$$

then *this sixth point* P' *can be derived from the five points* $\text{P}_1 \ldots \text{P}_5$ *by the same constructions, as those by which the point* $\text{P} = (x\,y\,z\,w\,v)$ *is derived from the five given points* A B C D E. For example, if we take the five points,

$$\text{A}_1 = (1\,0\,0\,0\,1), \quad \text{B}_1 = (0\,1\,0\,0\,1), \quad \text{C}_1 = (0\,0\,1\,0\,1), \quad \text{D}_1 = (0\,0\,0\,1\,1), \quad \text{E} = (0\,0\,0\,0\,1),$$

we have the symbolic equation,

$$(\text{A}_1) + (\text{B}_1) + (\text{C}_1) + (\text{D}_1) - 3(\text{E}) = (\text{U});$$

if then we write $v' = x + y + z + w - 3v$, the point $(x\,y\,z\,w\,v')$ is derived from $\text{A}_1\,\text{B}_1\,\text{C}_1\,\text{D}_1\,\text{E}$, by the same constructions as $(x\,y\,z\,w\,v)$ from A B C D E. In particular, D is related to $\text{A}_1\,\text{B}_1\,\text{C}_1\,\text{D}_1\,\text{E}$, as the point $\text{P} = (0\,0\,0\,3\,1)$ is related to A B C D E; but this point P satisfies the anharmonic equation, $(\text{D}\,\text{D}_1\,\text{E}\,\text{P}) = +3$; if then $\text{E}_1 = \text{D}_1\text{E} \cdot \text{A}_1\text{B}_1\text{C}_1 = (0\,0\,0\,\overline{1}\,2)$, we must have the corresponding equation $(\text{D}_1\text{E}_1\text{ED}) = +3$: which is accordingly found to exist and furnishes a *construction* for *exscribing a pyramid* ABCD *to a given pyramid* $\text{A}_1\text{B}_1\text{C}_1\text{D}_1$, with which it is to be *homologous*, and to have a *given point* E for the *centre* of their homology, agreeing with the construction assigned in [126.] for a similar problem of *exscription.* And in general, *from any five given points of a net*, whereof no four are complanar, we can (as was first shown by *Möbius*) *return, by linear constructions, to the five initial points* A … E; and therefore can, in this way, *reconstruct the net.*

---

* See page 295 of the *Barycentric Calculus.* As regards the theory of *homographic figures*, chapter XXV. of the *Géométrie Supérieure* of M. Chasles may be consulted with advantage. But with respect to *anharmonic ratio*, generally, it must be remarked that Professor *Möbius* was thoroughly familiar with its theory and practice, when he published in 1827; although he called it by the longer but perhaps more expressive name of Doppelschnittsverhältniss (*ratio bissectionalis*). It may be added that he denotes by (A, C, B, D), what I write as (A B C D).

[129.] If we content ourselves with *quaternary* (or *anharmonic*) *coordinates* [12.], or suppose (as we may) that $v = 0$, the *equation of a surface of the second order* takes the form,

$$0 = f(x\,y\,z\,w) = \alpha x^2 + \beta y^2 + \gamma z^2 + \delta w^2 + 2(\epsilon yz + \zeta zx + \eta xy) + 2w(\theta x + \iota y + \kappa z);$$

and if the ten coefficients $\alpha \ldots \kappa$, or their ratios, be determined by the condition that the surface shall pass *through nine given net-points*, those *coefficients* may then be replaced by *whole numbers*, and the *surface* may be said to be *rationally related to the given net*, or to the *initial system* A ... E, or briefly to be (comp. [8.]) a *Rational Surface*. For example, if the nine points be A B C D E C′ A′ C_2 A_2, so that, besides passing through E, the surface has the gauche quadrilateral ABCD superscribed upon it, the equation is

$$\text{I} \ldots \quad 0 = f = xz - yw;$$

and if they be A, B, A′, B′, A_2, B_2, A_1, $\mathrm{A}^{\mathrm{VII}} = (1\,2\,\bar{1}\,0)$, and $\mathrm{F} = (1\,2\,0\,\bar{1})$, so that this new point F, like $\mathrm{A}^{\mathrm{VII}}$, belongs to the group $\mathrm{P}_{2,6}$, the equation of the surface is then found to be,

$$\text{II} \ldots \quad 0 = f = w^2 + z^2 - (w+z)(x+y) - 2xy.$$

[130.] In general, whether the surface of the second order be *rational* or not, it results from the principles of a former communication that any point $\mathrm{P} = (x\,y\,z\,w)$ of space is the *pole* of the *plane* $\Pi = [X\,Y\,Z\,W]$, if $X \ldots W$ be the *derivatives*,

$$X = D_x f, \quad Y = D_y f, \quad Z = D_z f, \quad W = D_w f;$$

hence, in particular, the *pole of the plane* [E] *of homology* of the three pyramids E, E′, E‵, [26.] [113.] [125.], of which plane the *quaternary symbol* [12.] is [1 1 1 1], is the point K determined by the equations,

$$X = Y = Z = W, \quad \text{or} \quad D_x f = D_y f = D_z f = D_w f;$$

and if the point E be the mean point of the pyramid ABCD, the *plane* [E] is then *infinitely distant*, and this *point* K is the *centre of the surface.*

[131.] For example, in the case of the I[st] surface [129.], this *pole* K is the point $(1\,\bar{1}\,1\,\bar{1}) \equiv (2\,0\,2\,0\,1)$, which belongs to the group $\mathrm{P}_{3,1}$; and because it is *on* the plane [E], that plane *touches* the surface in that point: so that when the point E is the *mean* point of the pyramid ABCD, the surface becomes a ruled *paraboloid.* In the case of the II[nd] surface [129.], the pole K of [E] is always the point (1 1 0 0), or C′; this point C′ becomes therefore the *centre* of the surface, when E is the *mean* point of the pyramid; and the five following lines,

$$\mathrm{AB}, \quad \mathrm{A'B}_1^{\mathrm{VII}}, \quad \mathrm{B'A}^{\mathrm{VII}}, \quad \mathrm{A_2F}, \quad \text{and} \quad \mathrm{B_2G},$$

where G is the new point $(2\,1\,0\,\bar{1})$ of the group $\mathrm{P}_{2,6}$, which are *always chords through* C′, become in that case *diameters.* It may be added that, with the initial arrangement [91.], the surface last considered becomes the *sphere*, which is described with AB for diameter; and that it *always* passes through the auxiliary point P, of *third* construction, which was mentioned in [100.].

[132.] We have then here an *example*, of a surface of the second order, which was *determined* so as to pass [129.] through *nine net-points*

$$\text{A},\ \text{B},\ \text{A}',\ \text{B}',\ \text{A}_2,\ \text{B}_2,\ \text{A}_1,\ \text{A}^{\text{VII}},\ \text{and F},$$

but which has been subsequently *found* to pass *also* through at least *four other points of the net*, namely

$$\text{B}_1,\ \text{B}_1^{\text{VII}},\ \text{G},\ \text{and P}.$$

This is, however, only a very particular *case* of a much more general *Theorem*, with the enunciation of which I shall conclude the present Paper, regretting sincerely that it has already extended to a length, so much exceeding the usual limits of communications designed for the *Proceedings** of the Academy, but hoping that some at least of its processes and results will be thought not wholly uninteresting:—

"*If a Surface of the Second Order be determined by the condition of passing through nine given points of a Geometrical Net in Space, it passes through indefinitely many others: and every Point upon the Surface, which is not a point of the Net, can be included within a Geodetic Triangle on that surface, of which the corners are net-points, and of which the sides can be made as small as we may desire.*"

In fact, the *surface* is a *rational* one [129.], or the coefficients of its equation may be made whole numbers; and therefore *every rational line* [8.], from any *one net point*, or rational point, upon it, if not happening to *touch* the surface, is easily proved to meet it *again*, in *another rational point*: whence, with the aid of a lately mentioned principle [127.], the theorem evidently follows.

* *Some* of the early formulæ of this Paper are unavoidably repeated from a communication of the preceding Session (1859–60), but with extended significations, as connected now with a *quinary calculus*. And in a not yet published volume, entitled "*Elements of Quaternions*," the subject of *Nets in Space* is incidentally discussed, as an illustration of the *Method of Vectors*. But it will be found that the present Paper is far from being a mere reprint of the Section on Nets, in the unpublished work thus referred to: many new *theorems* having been introduced, and the *plan* of the treatment generally being different, although the *notations* have, on the whole, been retained. Besides it was thought that Members of the Academy might like to see the subject treated, in their Proceedings, without any express reference to *quaternions*: with which indeed the *nets* have not any *necessary* connexion.