

**ON THEOREMS RELATING TO SURFACES,  
OBTAINED BY THE METHOD OF QUATERNIONS**

**By**

**William Rowan Hamilton**

(Proceedings of the Royal Irish Academy, 4 (1850), pp. 306–308.)

Edited by David R. Wilkins

2000

*On Theorems relating to Surfaces, obtained by the Method of Quaternions.*

By Sir WILLIAM R. HAMILTON.

Communicated February 26, 1849.

[*Proceedings of the Royal Irish Academy*, vol. 4 (1850), pp. 306–308.]

The following letter from Sir William R. Hamilton was read, giving some general expressions of theorems relating to surfaces, obtained by his method of quaternions:

“The equation of curved surface being put under the form

$$f(\rho) = \text{const.} :$$

while its *tangent plane* may be represented by the equation,

$$df(\rho) = 0,$$

or

$$S \cdot \nu d\rho = 0,$$

if  $d\rho$  be the vector drawn to a point of that plane, from the point of contact; the equation of *an osculating surface of the second order* (having complete contact of the second order with the proposed surface at the proposed point) may be thus written:

$$0 = df(\rho) + \frac{1}{2}d^2f(\rho);$$

(by the extension of Taylor’s series to quaternions); or thus,

$$0 = 2S \cdot \nu d\rho + S \cdot d\nu d\rho,$$

if

$$df(\rho) = 2S \cdot \nu d\rho.$$

“The *sphere, which osculates in a given direction*, may be represented by the equation

$$0 = 2S \frac{\nu}{\Delta\rho} + S \frac{d\nu}{d\rho};$$

where  $\Delta\rho$  is a chord of the sphere, drawn from the point of osculation, and

$$S \frac{d\nu}{d\rho} = \frac{S \cdot d\nu d\rho}{d\rho^2} = \frac{d^2f(\rho)}{2d\rho^2}$$

is a scalar function of the versor  $U d\rho$ , which determines the direction of osculation. Hence the important formula:

$$\frac{\nu}{\rho - \sigma} = S \frac{d\nu}{d\rho};$$

where  $\sigma$  is the vector of the centre of the sphere which osculates in the direction answering to  $U d\rho$ .

“By combining this with the expression formerly given by me for a normal to the ellipsoid, namely

$$(\kappa^2 - \iota^2)^2 \nu = (\iota^2 + \kappa^2) \rho + \iota \rho \kappa + \kappa \rho \iota,$$

the known value of the curvature of a normal section of that surface may easily be obtained. And for *any* curved surface, the formula will be found to give easily this general theorem, which was perceived by me in 1824; that if, on a normal plane  $OPP'$ , which is drawn through a given normal  $PO$ , and through any linear element  $PP'$  of the surface, we project the infinitely near normal  $P'O'$ , which is erected to the same surface at the end of the element  $PP'$ ; the projection of the near normal will cross the given normal in the centre  $O$  of the same sphere which osculates to the given surface at the given point  $P$ , in the direction of the given element  $PP'$ .

“I am able to shew that the formula

$$0 = \delta S \frac{d\nu}{d\rho},$$

which follows from the above, for determining the directions of osculation of the greatest and least osculating spheres, agrees with my formerly published formula,

$$0 = S . \nu d\nu d\rho,$$

for the directions of the lines of curvature.

“And I can deduce Gauss’s *general* properties of geodetic lines by showing that if  $\sigma_1, \sigma_2$  be the two extreme values of the vector  $\sigma$ , then

$$\frac{-1}{(\rho - \sigma_1)(\rho - \sigma_2)} = \text{measure of curvature of surface} = \frac{1}{R_1 R_2} = \frac{d^2 T \delta \rho}{T \delta \rho . d\rho^2};$$

where  $d$  answers to motion along a normal section, and  $\delta$  to the passage from one near (normal) section to another; while  $S, T,$  and  $U,$  are the characteristics of the operations of taking the scalar, tensor and versor of a quaternion: and the variation  $\delta v$  of the inclination  $v$  of a given geodetic line to a variable normal section, obtained by passing from one such section to a near one, without changing the geodetic line, is expressed by the analogous formula,

$$\delta v = - \frac{dT \delta \rho}{T d\rho} .”$$