Note on Transformations of Posets with the Same Upper Bound Graph and Minimal Elements

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Abstract. Two posets with the same canonical poset and the same upper bound graph can be transformed into each other by a finite sequence of two kinds of transformations, called $x < y$-additions and $x < y$-deletions on minimal elements.

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1. Introduction

In this paper, we consider finite undirected simple graphs and finite posets. For a poset $P = (X, \leq)$, the upper bound graph (UB-graph) of $P = (X, \leq)$ is the graph $UB(P) = (X, E_{UB(P)})$, where $uv \in E_{UB(P)}$ if and only if $u \neq v$ and there exists $m \in X$ such that $u, v \leq m$. McMorris and Zaslavsky introduced this concept and gave a characterization of upper bound graphs [2].

Figure 1 shows two different posets which have the same upper bound graph. This example induces an interest in properties of posets with the same UB-graph.

In [3] and [4] we deal with sequences of transformations that convert a poset to any other poset that has the same upper bound graph. In this paper we show that the transformations can be of a special kind involving minimal elements of the posets at each step.

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2. Transformations of posets

For a poset \( P = (X, \leq) \) and \( x \in X \), \( L_P(x) = \{ y \in X ; y < x \} \) and \( U_P(x) = \{ y \in X ; y > x \} \). Furthermore \( V(P) \) is \( X \), \( \max(P) \) is the set of all maximal elements of \( P \), \( \min(P) \) is the set of all minimal elements of \( P \). For a poset \( P \) and \( x, y \in V(P) \), \( x \parallel_P y \) shows that \( x \) is incomparable with \( y \) in \( P \). For a poset \( P \), the canonical poset of \( P \) is the poset \( \text{can}(P) \) on the set \( V(P) \) in which \( x \leq_{\text{can}(P)} y \) if and only if (1) \( y \in \max(P) \) and \( x \leq_P y \), or (2) \( x = y \).

A clique in the graph \( G \) is the vertex set of a maximal complete subgraph. In some cases, we consider that a clique is a maximal complete subgraph. We say a family \( C \) of complete subgraphs edge covers \( G \) if for each edge \( uv \in E(G) \), there exists \( C \in C \) such that \( u, v \in C \).

**Theorem 2.1.** [2] Let \( G \) be a graph with \( n \) vertices. The graph \( G \) is a UB-graph if and only if there exists a family \( C = \{ C_1, C_2, ..., C_k \} \) of complete subgraphs of \( G \) such that

(a) \( C \) edge covers \( G \),

(b) for each \( C_i \), there exists a vertex \( v_i \in C_i - \bigcup_{j \neq i} C_j \).

Furthermore, such a family \( C \) must consist of cliques of \( G \) and is the only such family if \( G \) has no isolated vertices.

For a UB-graph \( G \) and an edge clique cover \( C = \{ C_1, C_2, ..., C_k \} \) satisfying the conditions of Theorem 2.1, a vertex subset \( K_{UB}(G) \) that consists of one element of each set \( C_i - \bigcup_{j \neq i} C_j \) is called a kernel of \( G \). We know a fact that, given any \( K_{UB}(G) \), there exists a poset \( P \) such that \( G = UB(P) \) and \( K_{UB}(G) = \max(P) \).

In the remainder of this paper, we consider a fixed labeled connected UB-graph \( G \) with a fixed kernel \( K_{UB}(G) \).

We define \( \mathcal{P}_{UB}(G) = \{ P ; UB(P) = G, \max(P) = K_{UB}(G) \} \). Each poset \( P \in \mathcal{P}_{UB}(G) \) is identified with the set of comparable pairs in \( P \). Thus \( \mathcal{P}_{UB}(G) \) is a poset by set inclusion. The canonical poset \( \text{can}(G) \) of \( G \) is the canonical poset of any poset \( P \in \mathcal{P}_{UB}(G) \). By Theorem 2.1, the canonical poset is independent of the choice. For a UB-graph \( G \), the canonical poset \( \text{can}(G) \) is a height 1 poset and \( V(\text{can}(G)) = \max(\text{can}(G)) \cup \min(\text{can}(G)) \).

To consider some relations among posets of \( \mathcal{P}_{UB}(G) \), we need some concepts as follows: For elements \( x \) and \( y \) in a poset \( P \) such that \( y \notin \max(P) \) and \( x \) is covered by \( y \), the poset \( P_{<y} \) is obtained from \( P \) by subtracting the relation \( x \leq y \) from \( P \), and we call this transformation the \( x < y \)-deletion. For an incomparable pair \( x \) and \( y \) in a poset \( P \) such that \( y \notin \max(P) \), \( U_P(y) \subseteq U_P(x) \) and \( L_P(y) \supseteq L_P(x) \), the poset \( P_{<y} \) is obtained from \( P \) by adding the relation \( x \leq y \) to \( P \), and we call this transformation the \( x < y \)-addition. We obtain the following facts on these transformations.

**Fact 1.** For a poset \( P \),

1. \( P \) and \( P_{<y} \) have the same UB-graph,
2. \( P \) and \( P_{<y} \) also have the same UB-graph, and
3. \( x < y \)-additions and \( x < y \)-deletions are inverse transformations to each other.
By these facts, we obtained the following result.

**Theorem 2.2.** [3] Let $G$ be a UB-graph and $P, Q$ be posets in $\mathcal{P}_{UB}(G)$. Then

1. $P$ can be transformed into $Q$ by a sequence of $x < y$-deletions and $x < y$-additions.
2. Every poset in $\mathcal{P}_{UB}(G)$ is obtained from $\text{can}(G)$ by $x < y$-additions only.

For an $x < y$-addition, we need to check two conditions: $U_P(y) \subseteq U_P(x)$ and $L_P(y) \supseteq L_P(x)$. If $x$ is a minimal element of a poset $P$, $L_P(x) = \emptyset$ and we only check the condition $U_P(y) \subseteq U_P(x)$ for an $x < y$-addition on an incomparable pair $x$ and $y$. The next result deals with $x < y$-additions on a minimal element $x$.

**Theorem 2.3.** Let $G$ be a UB-graph and $P$ be a poset in $\mathcal{P}_{UB}(G)$. The poset $P$ is obtained from $\text{can}(P)$ by $x < y$-additions only, where $x$ is a minimal element of the poset at each step.

**Proof.** Since $P$ is finite and every deletion reduces the number of comparable pairs in $P$, we obtain the following sequence of $x < y$-deletions, where $x$ is a minimal element of the poset of each step:

$$ P \xrightarrow{x < y - \text{deletion}} \ldots \xrightarrow{x < y - \text{deletion}} \text{can}(P) = \text{can}(G), $$

In each step of this sequence of $x < y$-deletions, $U_{P_{x < y}}(x) \supseteq U_{P_{x < y}}(y)$, $x$ is incomparable with $y$ in $P_{x < y}$ and a minimal element of $P_{x < y}$, and $y \notin \text{max}(P_{x < y})$. Thus the inverse operations of the above $x < y$-deletions are $x < y$-additions on $x, y$ satisfying that $x \parallel_P y$, $U_P(x) \supseteq U_P(y)$, $x$ is a minimal element of $P$ and $y$ is not a maximal element of $P$. So we obtain the following sequence of $x < y$-additions, where $x$ is a minimal element of the poset at each step.

$$ P \xleftarrow{x < y - \text{addition}} \ldots \xleftarrow{x < y - \text{addition}} \text{can}(P) = \text{can}(G). $$

From this result we obtain the next result.

**Theorem 2.4.** Let $G$ be a UB-graph and $P, Q$ be posets in $\mathcal{P}_{UB}(G)$. The poset $P$ can be transformed into $Q$ by a sequence of $x < y$-deletions and $x < y$-additions, where $x$ is a minimal element of the poset at each step and all the deletions can precede all the additions.

**Proof.** As in the proof of Theorem 2.3, $P$ can be reduced to $\text{can}(G)$ by $x < y$-deletions and then $\text{can}(G)$ can be enlarged to $Q$ by $x < y$-additions, where $x$ is always a minimal element of the poset at each step. 

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**References**

