In search for Lindelöf $C_p$’s  

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Abstract. It is shown that if $X$ is a first-countable countably compact subspace of ordinals then $C_p(X)$ is Lindelöf. This result is used to construct an example of a countably compact space $X$ such that the extent of $C_p(X)$ is less than the Lindelöf number of $C_p(X)$. This example answers negatively Reznichenko’s question whether Baturov’s theorem holds for countably compact spaces.

Keywords: $C_p(X)$, space of ordinals, Lindelöf space

Classification: 54C35, 54D20, 54F05

1. Introduction

We prove that $C_p(X)$ is Lindelöf for every first-countable countably compact subspace of ordinals. Thus, we widen the class of all spaces $X$ for which it is known that $C_p(X)$ is Lindelöf. This result gives some possible directions where one might find other spaces with Lindelöf $C_p$’s (see questions in Section 3). Using the main result we construct an example of a countably compact space $X$ such that $l(C_p(X)) \neq e(C_p(X))$. In the above equality $l(Y)$ stands for Lindelöf number, that is, the smallest infinite cardinal $\tau$ such that every open covering of $Y$ contains a subcovering of cardinality $\leq \tau$. And $e(Y)$ is the extent of $Y$ defined as the supremum of cardinalities of closed discrete subsets. This example answers Reznichenko’s question whether Baturov’s theorem [BAT] holds for countably compact spaces. Recall that Baturov’s theorem states that $l(Y) = e(Y)$ for every $Y \subset C_p(X)$, where $X$ is a $\Sigma$-Lindelöf space. A counterexample to Reznichenko’s question also answers negatively the question posed in [BUZ] whether $C_p(X)$ is a $D$-space if $X$ is countably-compact. The notion of $D$-space was introduced by Eric van Douwen [DOU].

A neighborhood assignment for a space $X$ is a function $\varphi$ from $X$ to the topology of $X$ such that $x \in \varphi(x)$ for any $x \in X$. A space $X$ is a $D$-space, if for any neighborhood assignment $\varphi$ for $X$ there exists a closed discrete subset $D$ of $X$ such that $X = \bigcup_{d \in D} \varphi(d)$.

Throughout the paper, all spaces are assumed to be Tychonov. By $R$ we denote the space of all real numbers endowed with standard topology. In notation and terminology we will follow [ARH] and [ENG].

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2. Main result

Let \( \tau_\omega = \{ \alpha \leq \tau : cf(\alpha) \leq \omega \} \). Since in this section we deal only with \( \tau_\omega \)'s and their function spaces, let us agree that for any \( \alpha, \beta \in (\tau + 1) \), by the interval \([\alpha, \beta]\) we mean the set \( \{ \gamma \in \tau_\omega : \alpha \leq \gamma \leq \beta \} \) (the same concerns open and half-open intervals). This agreement significantly simplifies our notation but is valid only within this section. If \( U \) is a standard open set of \( C_p(X) \) we say that \( U \) depends on a finite set \( \{x_1, \ldots, x_n\} \subset X \) if there exist \( B_1, \ldots, B_n \) open in \( R \) such that \( U = \{ f \in C_p(X) : f(x_i) \in B_i \text{ for } i \leq n \} \).

**Lemma 2.2.** If \( A \subset \tau_\omega \) is countable, then there exists an \( \omega \)-support \( B \) of \( A \).

**Proof:** For each \( a \in A \) non-isolated in \( \tau_\omega \), fix a countable strictly increasing sequence \( X_a \) of isolated ordinals converging to \( a \). Let \( B = A \cup \{0\} \cup (\bigcup_{a \in A} X_a) \).

The set \( B \) is countable as a countable union of countable sets. Conditions (1) and (2) are met by definition. Let us verify (3). Take any \( b \in B \) non-isolated in \( \tau_\omega \). Since all \( X_a \)'s consist of isolated ordinals, we have \( b \in A \). Therefore, \( b \) is an accumulation point for \( X_b \subset B \) and, as a consequence, for \( B \) as well.

Notice that if \( A_n \subset \tau_\omega \) is an \( \omega \)-support of itself for each \( n \), then \( \bigcup_n A_n \) is an \( \omega \)-support of itself as well.

**Definition 2.3.** Let \( A \subset \tau_\omega \) be countable and an \( \omega \)-support of itself. Let \( f \in C_p(\tau_\omega) \). Define \( cf,A \) as follows: \( cf,A(x) = f(a_x) \), where \( a_x = \sup\{a \in A : a \leq x\} \).

First notice that the set \( \{a \in A : a \leq x\} \) is not empty for every \( x \) because \( 0 \in A \) (see the definition of \( \omega \)-support). Since \( A \) is countable and \( \tau_\omega \) contains all ordinals not exceeding \( \tau \) of countable cofinality, \( a_x \) exists for each \( x \). And since the supremum is unique, \( cf,A \) is a well-defined function of \( \tau_\omega \) to \( R \). Also, notice that \( cf,A \) coincides with \( f \) on \( A \) as \( a_x = x \) for each \( x \in A \).

**Lemma 2.4.** Let \( A \subset \tau_\omega \) be countable and an \( \omega \)-support of itself. Let \( f \in C_p(\tau_\omega) \). Then \( cf,A \in C_p(\tau_\omega) \).

**Proof:** To show continuity of \( cf,A \) it is enough to show that for each \( x_n \rightarrow x \) in \( \tau_\omega \) one can find a subsequence \( \{x_{m_n}\} \subset \{x_n\} \) such that \( cf,A(x_{m_n}) \rightarrow cf,A(x) \). If \( x_n \in A \) for infinitely many of \( n \)'s then we are done since \( cf,A = f \) on \( A \).

Otherwise, we can assume that all \( x_n \)'s are not in \( A \) and are distinct. For each \( y \in \tau_\omega \), put \( b_y = \tau \) if \( \{y, \tau\} \cap A = \emptyset \) and \( b_y = \inf\{b \in A : b > y\} \) otherwise. For
each \(x_n\), consider \([a_{x_n}, b_{x_n}]\), where \(a_{x_n}\) is from the definition of \(c_{f,A}\). Notice that either \(b_y = \tau\) or \(b_{y'} = \tau\) is an isolated ordinal. Indeed, if \(b_y \neq \tau\) then \(b_{y'} = \inf\{b \in A : b > y\} \in A\). And since \(A\) is an \(\omega\)-support of itself, \(b_{y'}\) is an isolated ordinal (see condition (3) in Definition 2.1).

The intervals \([a_{x_n}, b_{x_n}]\) are either disjoint or coincide. Assume they coincide for infinitely many of \(m\)'s with \([a_{x_3}, b_{x_3}]\). If \(b_{x_3}\) is isolated then \(x \in [a_{x_3}, b_{x_3}]\) and \(c_{f,A}([a_{x_3}, b_{x_3}])\) is a singleton. Therefore, \(c_{f,A}(x_m) \rightarrow c_{f,A}(x)\). Otherwise \(b_{x_3}\) is not isolated and equal to \(\tau\). In this case \((a_{x_3}, \tau] \cap A = \emptyset\) and \(c_{f,A}([a_{x_3}, b_{x_3}])\) is a singleton again.

If the intervals are mutually disjoint then \(a_{x_n} \rightarrow x \in \bar{A}\). And now use the facts that \(f = c_{f,A}\) on \(\bar{A}\) and \(c_{f,A}(x_m) = f(a_{x_n})\).

\(\Box\)

Lemma 2.5. Let \(A \subseteq \tau_\omega\) be countable and an \(\omega\)-support of itself and \(B\) be a base of \(R\). Let \(f \in C_p(\tau_\omega)\). Let \(U \subset C_p(\tau_\omega)\) be open and contain \(c_{f,A}\). Then there exist sequences \(\{[a_1, b_1], \ldots, [a_n, b_n]\}\) and \(\{B_1, \ldots, B_n\}\) with the following properties:

\begin{enumerate}
  \item \(a_i \in A\);
  \item \(b_i \in A\) for \(i < n\) and \(b_n = \tau\);
  \item \(B_i \in B\);
  \item \(c_{f,A} \in \{g \in C_p(\tau_\omega) : g([a_i, b_i]) \subseteq B_i\text{ if }a_i \neq b_i\text{ and }g(a_i) \in B_i\text{ if }a_i = b_i\}\).
\end{enumerate}

\textbf{Proof:} Without loss of generality, there exist \(c_1 < \ldots < c_i \in \tau_\omega\) and \(V_1, \ldots, V_i \in B\) such that \(U = \{g \in C_p(\tau_\omega) : g(c_i) \in V_i\}\). We may assume that \(c_1 \geq \sup(\bar{A})\).

\textbf{Step 1.}

Let \(m = \min\{i : c_i \geq \sup(\bar{A})\}\). Find \(B_1 \in B\) such that \(c_{f,A}(c_m) \in B_1 \subset V_m \cap V_{m+1} \cap \cdots \cap V_i\). Note that such a \(B_1\) exists since \(c_{f,A}\) is constant starting from \(\sup(\bar{A})\). Find \(a_1 \in A\) such that \(c_{f,A}([a_1, \tau]) \subseteq B_1\) and \(a_1 > c_i\) for all \(i < m\). Due to continuity of \(c_{f,A}\), such an \(a_1\) can be found somewhere close to \(\sup(\bar{A})\) (if \(\sup(\bar{A}) \in A\), it can serve as \(a_1\)). Put \(b_1 = \tau\).

\textbf{Step \(k \leq 1\).}

If \(c_i \geq a_{k-1}\) for all \(i\), stop construction. Let \(m = \max\{i : c_i < a_{k-1}\}\). Let \(a'_k = \sup\{a \in A : a \leq c_m\}\) and \(b_k = \inf\{a \in A : c_m \leq a\}\). Obviously \(b_k \in A\). If \(b_k = c_m = a'_k\) put \(a_k = c_m\) and \(B_k = V_m\). Otherwise, find \(B_k \in B\) such that \(c_{f,A}([a'_k, b_k]) \subset B_k \subset V_m\). Such a \(B_k\) exists because \(c_{f,A}([a'_k, b_k]) = f(a'_k) = c_{f,A}(c_m)\). If \(a'_k = c_m-1\) we also require that \(B_k \subset V_m \cap V_{m-1}\). If \(a'_k \in A\) put \(a_k = a'_k\). Otherwise \(a'_k\) is an accumulation point for \(A\). And, due to continuity, we can find an \(a_k \in A\) such that \([a_k, a'_k]\) contains no \(c_i\)'s and \(c_{f,A}([a_k, b_k]) \subset B_k\).

Re-enumerate \(B_1, \ldots, B_n\) and corresponding intervals in reverse order. Properties (1)--(4) hold by our construction. \(\Box\)
Theorem 2.6. \( C_p(\tau_\omega) \) is Lindelöf for any \( \tau \).

Proof: Let \( B \) be a countable base of \( R \). Let \( \mathcal{U} \) be an arbitrary open covering of \( C_p(\tau_\omega) \). We will choose a countable subcovering \( \{ U_n \} \) inductively. From Step 2, we will follow our induction using elements in \( \mathcal{S}_1 \) defined at Step 1. However, at each Step \( n \) we might need to enlarge our inductive set by new elements. To ensure that every old element keeps the old tag we agree to enumerate \( \mathcal{S}_1 \) by prime numbers while new elements added at Step \( n \) by numbers \( p^{n+1} \), where \( p \) is any prime.

Step 1.

Take any \( U_1 \in \mathcal{U} \). The set \( U_1 \) depends on finite \( X_1 \). Let \( A_1 \) be an \( \omega \)-support of \( X_1 \). Let \( \mathcal{S}_1 \) consist of all pairs \( ([a_1, b_1], \ldots, [a_k, b_k]), \{ B_1, \ldots, B_k \}) \), where \( B_i \in B, a_i \in A_1, b_i \in A_1 \) for \( i < k \), \( b_k = \tau \), and \( k \) is any natural number. Enumerate \( \mathcal{S}_1 \) by prime numbers.

Step \( n \).

If \( U_1 \cup \cdots \cup U_{n-1} \) covers \( C_p(\tau_\omega) \) stop induction. Otherwise, take the first \( S = ([a_1, b_1], \ldots, [a_k, b_k]), \{ B_1, \ldots, B_k \}) \in \mathcal{S}_{n-1} \) such that there exist \( f \) and \( U_n \in \mathcal{U} \) containing \( f \) and the following property is satisfied.

Property. \( f \in \{ g \in C_p(\tau_\omega) : g([a_i, b_i]) \subset B_i \text{ if } a_i \neq b_i \text{ and } g(a_i) \in B_i \text{ if } a_i = b_i \} \subset U_n. \)

If no such an \( S \) exists, just take any \( U_n \in \mathcal{U} \) such that \( U_n \setminus \bigcup_{i<n} U_i \neq \emptyset. \)
The set \( U_n \) depends on \( X_n \). Let \( A_n \) be an \( \omega \)-support of \( A_{n-1} \cup X_n \). Let \( \mathcal{S}_n \) be the set of all pairs \( ([a_1, b_1], \ldots, [a_k, b_k]), \{ B_1, \ldots, B_k \}) \), where \( B_i \in B, a_i \in A_n, b_i \in A_n \) for \( i < k \), \( b_k = \tau \), and \( k \) is any natural number. Enumerate \( \mathcal{S}_n \setminus \mathcal{S}_{n-1} \) by numbers \( p^{n+1} \), where \( p \) is any prime number. Enumeration on \( \mathcal{S}_{n-1} \) is left unchanged.

Let us show that \( \bigcup_n U_n \) covers \( C_p(\tau_\omega) \). Take any \( f \in C_p(\tau_\omega) \). Let \( A = \bigcup_n A_n \). The set \( A \) is an \( \omega \)-support of itself. Consider the function \( c_{f,A} \). Since \( \mathcal{U} \) covers \( C_p(\tau_\omega) \) there exists \( U \in \mathcal{U} \) that contains \( c_{f,A} \).

By Lemma 2.5, there exists a pair \( S = ([a_1, b_1], \ldots, [a_k, b_k]), \{ B_1, \ldots, B_k \}) \) with the following properties:

1. \( a_i \in A; \)
2. \( b_i \in A \) for \( i < k \) and \( b_k = \tau; \)
3. \( B_i \in B; \)
4. \( c_{f,A} \in \{ g \in C_p(\tau_\omega) : g([a_i, b_i]) \subset B_i \text{ if } a_i \neq b_i \text{ and } g(a_i) \in B_i \text{ if } a_i = b_i \} \subset U. \)

That is, \( S \in \mathcal{S}_n \) for some \( n \). Therefore, starting from some Step \( p^{n+1} \), \( S \) must satisfy the Property and eventually it will be the first such. Therefore, \( c_{f,A} \) must be covered by some \( U_m \) chosen at Step \( m \). However, \( U_m \) depends on \( X_m \subset A_m \subset A \) while \( c_{f,A} \) coincides with \( f \) on \( A \). Therefore, \( U_m \) covers \( f. \)
Since any first-countable countably compact subspace of ordinals is homeomorphic to $\tau_\omega$ for some $\tau$ we can restate our result as follows.

**Theorem 2.7.** Let $X$ be a first-countable countably compact subspace of ordinals. Then $C_p(X)$ is Lindelöf.

3. Corollaries and related questions

Many papers are devoted to finding classes of spaces with Lindelöf $C_p$’s. How good a space should be to have such a nice covering property as Lindelöfness in its function space? It is known that even a linearly orderable first countable compactum is not such unless it is metrizable. This fact follows from the theorem of Nahmanson in [NAH] (a detailed proof is in [ARH]). His theorem states that if $X$ is a linearly ordered compactum then the Lindelöf number of $C_p(X)$ equals the weight of $X$. Even first-countable compacta with metrizable closures of countable sets do not have to have Lindelöf $C_p$’s. Again this follows from the Nahmanson theorem and existence of non-metrizable first countable linearly ordered compacta in which closures of countable sets are metrizable (an example of such a compactum is Aronszajn continuum).

However, what happens if we strengthen the requirement of metrizable closures to countable closures? Notice that spaces in our main result (Theorem 2.7) are first-countable countably compact and, the closures of countable sets are countable. Therefore, the following questions might be of interest.

**Question 3.1.** Let $X$ be countably compact and first countable. Assume also that the closure of any countable set is countable in $X$. Is then $C_p(X)$ Lindelöf?

**Question 3.2.** Let $X$ be first-countable and countably compact. Assume also that the closure of any countable set is countable in $X$. Is then $C_p(X)^\omega$ Lindelöf?

**Question 3.3.** Let $X = X_1 \oplus \cdots \oplus X_n \oplus \ldots$, where each $X_n$ is first-countable and countably compact. Assume also that the closure of any countable set is countable in $X_n$. Is then $C_p(X)$ Lindelöf?

Notice that spaces in Question 3.3 can be obtained from spaces in Question 3.1 by removing a point of countable character. Therefore the following question might worth consideration.

**Question 3.4.** Suppose that $C_p(X)$ is Lindelöf for a space $X$. Let $x \in X$ have countable character in $X$. Is $C_p(X \setminus \{x\})$ Lindelöf? What if $X$ is first countable (countably compact)?

So we throw away a point and are hoping that what is left still has a decent $C_p$. Why do we not add one point? In general, adding a point can spoil $C_p$. For example, $C_p(\omega_1)$ is Lindelöf by Theorem 2.7, while $C_p(\omega_1 + 1)$ is not by Asanov’s theorem [ASA]. Asanov’s theorem implies that if $C_p(X)$ is Lindelöf then the tightness of $X$ is countable (the tightness $t(X)$ of a space $X$ is the smallest infinite
cardinal number $\tau$ such that for any $A \subset X$ and any $x \in A$ there exists $B \subset A$ of cardinality not exceeding $\tau$ such that $x \in B$). That is, by adding one point \{\omega_1\} we lose Lindelöfness of the function space. This observation motivates the following question.

**Question 3.5** (Arhangelskii). Let $C_p(X \setminus \{x\})$ be Lindelöf and let $x$ have countable tightness in $X$. Is $C_p(X)$ Lindelöf? What if $X$ is first countable?

Our next corollary is an answer to the Reznichenko’s question whether Baturov’s theorem [BAT] holds for countably compact spaces. Baturov’s theorem states that $l(Y) = e(Y)$ for every $Y \subset C_p(X)$, where $X$ is a $\Sigma$-Lindelöf space.

We answer Reznichenko’s question by constructing a countably compact space $X$ where the above equality fails to hold.

In the following example, by $[\alpha, \beta]_X$ we denote the set $[\alpha, \beta] \cap X$, where $\alpha, \beta \in \tau$ and $X \subset \tau$.

**Example 3.6.** Let $X = \{\alpha \leq \omega_2 : cf(\alpha) \neq \omega_1\}$. Then $l(C_p(X)) = \omega_2$ while $e(C_p(X)) = \omega$.

**Proof of $e(C_p(X)) = \omega$:**

It suffices to show that any $F \subset C_p(X)$ of cardinality $\omega_1$ has a complete accumulation point in $C_p(X)$. Due to cofinality, there exists $\gamma < \omega_2$ such that $f$ is constant on $[\gamma, \omega_2)_X$ for each $f \in F$. We can also choose $\gamma$ with countable cofinality. For each $f \in F$ let $f^* \in C_p(\gamma, \omega)$ be such that $f^* = f$ on $[0, \gamma)_{\gamma, \omega}$. Since $C_p(\gamma, \omega)$ is Lindelöf (Theorem 2.6), there exists $h^* \in C_p(\gamma, \omega)$ a complete accumulation point for $F^* = \{f^* : f \in F\}$. Define a function $h$ as follows:

$$h(x) = \begin{cases} h^*(x) & \text{if } x \in [0, \gamma)_X, \\ h^*(\gamma) & \text{if } x \in [\gamma, \omega_2)_X. \end{cases}$$

No doubts, $h \in C_p(X)$. Let us show that $h$ is a complete accumulation point for $F$. Let $h \in U = \{g \in C_p(X) : g(c_i) \in B_i\}$, where $c_1 < \cdots < c_n \in X$ and $B_1, \ldots, B_n$ are open in $R$. We need to show that $F \cap U$ is uncountable. It does not hurt if we make $U$ smaller by assuming that $c_k = \gamma$ for some $k \leq n$. Since $h$ is constant on $[\gamma, \omega_2)_X$ we may assume that $B_j = B_k$ for all $j \geq k$.

The set $U^* = \{g \in C_p(\gamma, \omega) : g(c_i) \in B_i, i \leq k\}$ is an open neighborhood of $h^*$. Since $h^*$ is a complete accumulation point for $F^*$, $F^* \cap U^*$ is uncountable. If $f^* \in U^* \cap F^*$ then $f^*(c_k) \in B_k$. Therefore, for $j > k$, $f(c_j) = f(c_k) \in B_j$ and $f(c_j) \in B_j$ for $j \leq k$ because $f$ coincides with $f^*$ on $[0, \gamma)_X = [0, \gamma)_{\gamma, \omega}$. Therefore, $f \in F \cap U$ and $f \cap U$ is uncountable.

**Proof of $l(C_p(X)) = \omega_2$:**

Asanov’s theorem [ASA] implies that $t(X) \leq l(C_p(X))$. Since $t(X) = \omega_2$, $l(C_p(X)) \geq \omega_2$. And we actually have equality because the weight of $X$ is $\omega_2$. \qed
In [BUZ], the author proves that $C_p(X)$ is hereditarily a $D$-space if $X$ is compact. This result motivated the $D$-version of Reznichenko’s question whether $C_p(X)$ is a hereditary $D$-space if $X$ is countably compact. From the definition of a $D$-space it is easy to conclude that $l(X) = e(X)$ for every $D$-space $X$. Therefore, Example 3.6 serves as a counterexample to this question.

One of the central questions on $D$-spaces posed by van Douwen is whether every Lindelöf space is a $D$-space. In search for a counterexample (if there exists one) it might be worth to consider the following question.

**Question 3.7.** Is $C_p(\tau_\omega)$ a $D$-space for $\tau \geq \omega_2$?

Note that all theorems on $D$-spaces known so far do not cover the spaces in the above question.

**After-Submission Remarks.** After this paper was submitted, A. Dow and P. Simon answered Question 3.1 in negative. Therefore, it is reasonable to assume now that $C_p(X)$ in Question 3.2 and $C_p(X_n)$’s in Question 3.3 are Lindelöf.

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**References**


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