Recursively differentiable quasigroups
and complete recursive codes

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Abstract. Criteria of recursive differentiability of quasigroups are given. Complete recursive codes which attains the Joshi bound are constructed using recursively differentiable k-ary quasigroups.

Keywords: k-recursive code, strong orthogonal system of quasigroups, recursively differentiable quasigroups.

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Let q, n be positive integers and Q be a nonempty set of q elements. A code C ⊆ Q^n of length n over the alphabet Q is called an [n, k]_Q-code if |C| = q^k. An [n, k, d]_Q-code is a [n, k]_Q-code with the minimal Hamming distance d [1].

According to D.D. Joshi’s theorem [2], if C is an [n, k, d]_Q-code, then |C| ≤ q^{n−d+1}, where |Q| = q.

If an [n, k, d]_Q-code C has the cardinal number |C| = q^{n−d+1} then we say that C attains the Joshi bound. The problem of description of the parameters q, n and d for which there exist [n, k, d]_Q-codes, where |Q| = q, attaining the Joshi bound is open [1].

It is known that using strong orthogonal systems of k-ary quasigroups (k ≥ 2), in particular, orthogonal systems of latin squares, such codes can be constructed.

For example, if \{f_1, f_2, \ldots, f_t\}, t ≥ 2, is an orthogonal system of binary quasigroups defined on a set Q of q elements, then

\[ C = \{(x, y, f_1(x, y), f_2(x, y), \ldots, f_t(x, y)) \mid x, y \in Q\} \]

is an [t + 2, 2, t + 1]_Q-code, so C attains the Joshi bound [2].

This article deals with complete k-recursive codes and recursive differentiability of k-ary quasigroups.

A code C of length n over an alphabet Q is called complete k-recursive, where 1 ≤ k ≤ n, if there exists a mapping f : Q^k → Q such that every code word \( u = (u_0, u_1, \ldots, u_{n−1}) \in C \) satisfies the conditions

\[ u_{i+k} = f(u_i, u_{i+1}, \ldots, u_{i+k−1}), \]
for every \( i = 0, 1, \ldots, n - k \).

A complete \( k \)-recursive code \( C \subseteq Q^n \) defined by the mapping \( f \) is denoted by \( C(n, f) \).

In what follows we will use the notation \((x_1^k)\) for \((x_1, \ldots, x_k)\).

It is proved in [1] and it is easy to see that if \( C(n, f) \) is a complete \( k \)-recursive code over an alphabet \( Q \) then

\[
C(n, f) = \{(x_1, \ldots, x_k, f^{(0)}(x_1^{k-1}), \ldots, f^{(n-k-1)}(x_1^{k})) \mid x_1, \ldots, x_k \in Q\},
\]

where the functions \( f^{(0)}, f^{(1)}, \ldots, f^{(n-k-1)} \) are called \( k \)-recursive derivatives of \( f \) and are defined as follows:

\[
f^{(0)}(x_1^k) = f(x_1^k), \quad f^{(1)}(x_1^k) = f(x_2^k, f^{(0)}(x_1^k)) ,
\]

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\[
f^{(t)}(x_1^k) = f(x_{t+1}^k, f^{(0)}(x_1^k), f^{(1)}(x_1^k), \ldots, f^{(t-1)}(x_1^k)) , \text{ for } t < k,
\]

\[
f^{(t)}(x_1^k) = f(f^{(t-k)}(x_1^k), \ldots, f^{(t-1)}(x_1^k)) , \text{ for } t \geq k.
\]

A \( k \)-ary quasigroup operation \( f \) \((k \geq 2)\) is called recursively \( s \)-differentiable if its \( k \)-recursive derivatives \( f^{(0)}, f^{(1)}, \ldots, f^{(s)} \) are \( k \)-ary quasigroup operations. Let \( k \in \mathbb{N}, k \geq 2, \) and let \( f_1, f_2, \ldots, f_k \) be \( k \)-ary operations defined on a set \( Q \).

The operations \( f_1, f_2, \ldots, f_k \) are called orthogonal if the system of equations \( \{ f_i(x_1, x_2, \ldots, x_k) = a_i \}_{i=1}^k \) has a unique solution for every \( a_1, \ldots, a_k \in Q \). It is known and it is easy to see that the \( k \)-ary operations \( f_1, f_2, \ldots, f_k \), defined on a set \( Q \) are orthogonal if and only if the mapping

\[
\theta : Q^k \to Q^k , \quad \theta(x_1^k) = (f_1(x_1^k), f_2(x_1^k), \ldots, f_k(x_1^k)) = (f_1, f_2, \ldots, f_k)(x_1^k)
\]

is a bijection. In this case we will denote \( \theta = (f_1, f_2, \ldots, f_k) \).

A system \( \Sigma = \{ f_1, f_2, \ldots, f_i \}_{i \geq k} \) of \( k \)-ary operations defined on a set \( Q \) is called orthogonal if every \( k \) operations from \( \Sigma \) are orthogonal. A system \( \{ f_1, f_2, \ldots, f_s \}_{s \geq 1} \) of \( k \)-ary operations defined on a set \( Q \) is called strong orthogonal if the system \( \{ E_1, \ldots, E_k, f_1, f_2, \ldots, f_s \} \) is orthogonal, where \( E_i(x_1^k) = x_i \), for every \((x_1, \ldots, x_k) \in Q^k \) and for every \(i = 1, 2, \ldots, k \) (the \( k \)-ary selectors).

It follows from the definition that each operation of a strong orthogonal system, which is not a selector, is a quasigroup operation. Every orthogonal system of binary quasigroups is strong orthogonal.

It is proved in [1] that a complete \( k \)-recursive code \( C(n, f) \) attains the Yoshibound if and only if the system of \( k \)-recursive derivatives \( \{ f^{(0)}, f^{(1)}, \ldots, f^{(n-k-1)} \} \) is strong orthogonal. In this case the \( k \)-recursive derivatives \( f^{(0)}, f^{(1)}, \ldots, f^{(n-k-1)} \) of \( f \) are \( k \)-ary quasigroup operations, so \( f \) is recursively \((n-k-1)\)-differentiable. The converse is not true for \( k \geq 3 \). But for \( k = 2 \) the following criterion holds.
**Proposition 1** ([1]). A complete 2-recursive code

\[ C(n, f) = \{(x, y, f^{(0)}(x, y), f^{(1)}(x, y), \ldots, f^{(n-3)}(x, y)) \mid x, y \in Q\} \]

attains the Joshi bound if and only if the 2-recursive derivatives \( f^{(0)}, f^{(1)}, \ldots, f^{(n-3)} \) of \( f \) are quasigroup operations.

So a complete 2-recursive code \( C(n, f) \) attains the Joshi bound if and only if the binary operation \( f \) is recursively \((n - 3)\)-differentiable.

As was announced by G. Belyavskaya in [7] if \( Q(f) \) is a binary quasigroup then

\[ f^{(i)} = f^{\theta^i}, \forall i \in \mathbb{N}, \text{ where } \theta \text{ is the following mapping:} \]

\[ \theta : Q^2 \longrightarrow Q^2, \theta(x, y) = (y, f(x, y)), \forall (x, y) \in Q^2. \]

So Proposition 1 has the following algebraic meaning: a binary quasigroup \( Q(f) \) is recursively \( s \)-differentiable \((s \in \mathbb{N})\) if and only if \( f, f\theta, \ldots, f^{s-1}\theta \), where \( \theta = (E_2, f) \), are quasigroup operations. The result announced in [7] is generalized in the following proposition.

**Proposition 2.** If \( f \) is a \( k \)-ary operation \((k \geq 2)\) then \( f^{(n)} = f^{\theta^n} \) for all \( n \in \mathbb{N} \), where

\[ \theta : Q^k \longrightarrow Q^k, \theta(x_1, \ldots, x_k) = (x_2, \ldots, x_k, f(x_1)) \]

for every \((x_1^k) \in Q^k\).

**Proof:** To prove this proposition we will use the mathematical induction.

For \( n = 0 \) and \( n = 1 \), according to the definition of \( k \)-recursive derivatives, we have \( f^{(0)} = f = f^{\theta^0} \) and \( f^{(1)} = f(E_2, \ldots, E_k, f) = f\theta \).

Let us suppose that Proposition 2 is true for every \( n \), satisfying the inequalities:

\[ 0 \leq n \leq s - 1 < k. \]

Then for \( n = s \), using this assumption, we get:

\[ f^{(s)} = f(E_{s+1}, \ldots, E_k, f^{(0)}, \ldots, f^{(s-1)}) = f(E_{s+1}, \ldots, E_k, f, f\theta, \ldots, f\theta^{s-1}) \]

\[ = f(E_s, \ldots, E_k, f\theta, \ldots, f\theta^{s-2})\theta = f\theta^{s-1}\theta = f\theta^s. \]

For \( n = k \) have

\[ f^{(k)} = f(f^{(0)}, f^{(1)}, \ldots, f^{(k-1)}) = f(E_k, f^{(0)}, f^{(1)}, \ldots, f^{(k-2)})\theta = f\theta^{k-1}\theta = f\theta^k. \]

Let us suppose now that Proposition 2 is true for every \( n \leq m - 1 \), where \( m \geq k + 1 \). Then

\[ f^{(m)} = f(f^{(m-k)}, \ldots, f^{(m-2)}, f^{(m-1)}) \]

\[ = f(f^{(m-k-1)}, \ldots, f^{(m-3)}, f^{(m-2)})(E_2, \ldots, E_k, f) = f\theta^{m-1}\theta = f\theta^m. \]

So Proposition 2 is true for every \( n \in \mathbb{N} \). □
Corollary. Let $Q(f)$ be an $k$-ary quasigroup, $k \geq 2$ and $s \in \mathbb{N}$. If $\{f, f\theta, \ldots, f\theta^s\}$, where $\theta$ is the mapping defined in (1), is a strong orthogonal system of $k$-ary operations then $Q(f)$ is recursively $s$-differentiable.

As was shown above for $k = 2$ the converse of this corollary is true as well.

**Proposition 3.** Let $Q(f)$ be an $k$-ary quasigroup, $k \geq 2$. Every $k+1$ consecutive $k$-recursive derivatives $\{f^{(i)}, f^{(i+1)}, \ldots, f^{(i+k)}\}$ of $f$ are orthogonal.

**Proof:** If $Q(f)$ is an $k$-ary quasigroup, $k \geq 2$, then the system $\Sigma = \{E_1, \ldots, E_k, f\}$ is orthogonal, so its subsystem $\{E_2, \ldots, E_k, f\}$ is orthogonal as well, i.e. the mapping

$$\theta : Q^k \longrightarrow Q^k, \quad \theta(x^k) = (x_2, \ldots, x_k, f(x^k)), \quad \forall (x^k) \in Q^k,$$

is a bijection. Hence each of the following systems is orthogonal:

$$\Sigma^2 = \{E_3, \ldots, E_k, f, f\theta, f\theta^2\} = \{E_3, \ldots, E_k, f^{(0)}, f^{(1)}, f^{(2)}\}, \ldots,$$

$$\Sigma^{k-1} = \{E_k, f, f\theta, \ldots, f\theta^{k-1}\} = \{E_k, f^{(0)}, f^{(1)}, \ldots, f^{(k-1)}\},$$

$$\Sigma^k = \{f, f\theta, \ldots, f\theta^k\} = \{f^{(0)}, f^{(1)}, \ldots, f^{(k)}\}$$

and

$$\Sigma = \{f^{\theta^s-k}, \ldots, f^{\theta^s}\} = \{f^{(s-k)}, \ldots, f^{(s)}\},$$

for every $s \geq k + 1$. \qed

**Corollary 1.** A binary quasigroup $Q(f)$ is recursively 1-differentiable if and only if the pair of operations $\{E_1, f^{(1)}\}$ is orthogonal.

**Proof:** As $\{E_1, E_2, f\}$ is an orthogonal system, the mapping $\theta = (E_2, f)$ is a bijection and the system $\{E_2, f^{(1)}\} = \{E_1, E_2, f\} \theta$ is orthogonal too. Hence, $f^{(1)}$ is a quasigroup operation if and only if the pair $\{E_1, f^{(1)}\}$ is orthogonal. \qed

**Corollary 2.** A ternary quasigroup $Q(f)$ is recursively 1-differentiable iff the systems of ternary operations $\{E_1, E_2, f^{(1)}\}$ and $\{E_1, E_3, f^{(1)}\}$ are orthogonal.

Let $Q(\cdot)$ be a binary group and let denote by $\langle \Delta \rangle$ the $n$-th 2-recursive derivative of $\cdot$, for every $n \in \mathbb{N}$.
Lemma 1. If $Q(\cdot)$ is an abelian group, then for all $x, y \in Q$ and $n \in \mathbb{N}$ the following equality holds:

\[ x \triangle y = x^{b_n} y^{b_{n+1}} \]

where $(b_n)_{n \in \mathbb{N}}$ is the Fibonacci sequence.

**Proof:** We will use the mathematical induction.

For $n = 0$ have $x \triangle y = x \cdot y$ so $0^0 = x^{b_0} y^{b_1}$.

For $n = 1$ have $x \triangle y = y \cdot xy = x \cdot y^2 = x^{b_1} y^{b_2}$.

Suppose that Lemma 1 is true for every $n \leq k$. Using this assumption and the definition of the Fibonacci sequence, for $n = k + 1$ we get

\[
\begin{align*}
    k+1 \quad x \triangle y &= (x \triangle y)(x \triangle y) = x^{b_{k-1}} y^{b_k} x^{b_k} y^{b_{k+1}} \\
    &= x^{b_{k-1} + b_k} y^{b_k + b_{k+1}} = x^{b_{k+1}} y^{b_{k+2}}.
\end{align*}
\]

So the equality (2) is true for every $x, y \in Q$ and for every $n \in \mathbb{N}$. \[
\square
\]

**Theorem 1.** A binary abelian group $Q(\cdot)$ is recursively $s$-differentiable, where $s \geq 1$, if and only if the mappings $x \mapsto x^b$, where $(b_n)_{n \in \mathbb{N}}$ is the Fibonacci sequence, are bijections for all $i \in \{0,1,2,\ldots,s+1\}$.

**Proof:** According to the definition a group $Q(\cdot)$ is recursively $s$-differentiable if and only if its $2$-recursive derivatives $(\triangle), (\triangle), \ldots, (\triangle)$ are quasigroup operations. Hence $Q(\cdot)$ is recursively $s$-differentiable if and only if each of the equations $x \triangle a = c$, $a \triangle y = c$, $i \in \{0,1,2,\ldots,s\}$, has a unique solution for every $a, c \in Q$. Now, using the equalities (2) we get: $x_k \triangle a = c \iff x_{k-1} \cdot a^{b_k} = c \iff x^{b_k} = c \cdot a^{-b_k+1}$ and $a \triangle y = c \iff y_{k+1} \cdot a^{b_k} = c \iff y^{b_k+1} = c \cdot a^{-b_k}$ for every $a, c \in Q$ and for every $i \in \{0,1,2,\ldots,s\}$. So $(\triangle), (\triangle), \ldots, (\triangle)$ are quasigroup operations if and only if the mappings $x \mapsto x^b$, where $(b_i)_{i \in \mathbb{N}}$ is the Fibonacci sequence, are bijections for every $i \in \{0,1,2,\ldots,s+1\}$. \[
\square
\]

**Proposition 4.** If $Q(\cdot)$ is an arbitrary recursively $s$-differentiable binary group, where $s \geq 1$, then the mappings $x \mapsto x^b$, where $(b_i)_{i \in \mathbb{N}}$ is the Fibonacci sequence, are bijections for all $i \in \{0,1,2,\ldots,s+1\}$.

**Proof:** If $Q(\cdot)$ is recursively $s$-differentiable, with unit $e$, then each of the equations $e \triangle x = c$ and $y \triangle e = c$, $i \in \{0,1,2,\ldots,s\}$, has a unique solution. So as
When $s = 1$ Theorem 1 is true for an arbitrary binary group as we can see from the following proposition.

**Proposition 5.** A binary group $Q(\cdot)$ is recursively 1-differentiable if and only if the mapping $z \mapsto z^2$ is a bijection.

**Proof:** According to the definition, a binary group $Q(\cdot)$ is recursively 1-differentiable if and only if its 2-recursive derivative $(\triangle)$ is a quasigroup operation. So as

$$a \triangle x = b \iff x \cdot ax = b \iff xaxa = ba \iff (xa)^2 = ba,$$

for every $a, b \in Q$, we get that the mapping $z \mapsto z^2$ is a bijection if and only if the equation $a \triangle (za^{-1}) = b$ has a unique solution $z$ for every $a, b \in Q$.

From the equivalences $x \triangle a = b \iff a \cdot xa = b \iff x = a^{-1}ba^{-1}$ it follows that in a binary quasigroup $Q(\cdot)$ the equation $x \triangle a = b$ has always a unique solution for every $a, b \in Q$. So if $Q(\cdot)$ is a group then $Q(\triangle)$ is a quasigroup if and only if the mapping $z \mapsto z^2$ is a bijection. 

**Corollary.** A finite binary group is recursively 1-differentiable if and only if it is of odd order.

Indeed, it is known [3] that a finite group is of odd order if and only if the mapping $z \mapsto z^2$ is a bijection.

**Proposition 6.** If $Q(\cdot)$ is a binary group with unit $e$, then $Q(\triangle)$ is a semigroup if and only if $x^2 = e$, for every $x \in Q$.

**Proof:** So as $(x\triangle y) \triangle z = zyxzxz$ and $x\triangle (y\triangle z) = zyxzxzyz$, for all $x, y, z \in Q$, we get that the operation $(\triangle)$ is associative if and only if $x = zxz$, for every $x, z \in Q$. Taking $x = e$ in the last equality we get $z^2 = e$, for all $z \in Q$. Conversely, if $x^2 = e$, for all $x \in Q$, then $x = x^{-1}$ and $zx \cdot xz = e$, $\forall x, z \in Q$, so $zxz = x^{-1} = x$, for all $x, z \in Q$, i.e. $(\triangle)$ is associative.

**Corollary.** If $Q(\cdot)$ is a nontrivial recursively 1-differentiable group then its 2-recursive derivative $Q(\triangle)$ cannot be a group.

**Proof:** Indeed, if $Q(\cdot)$ is recursively 1-differentiable and $Q(\triangle)$ is a group, then according to Proposition 5, the mapping $z \mapsto z^2$ is a bijection and by Proposition 6 we get $|Q| = 1$. 

\[\Box\]
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