Note on the classification theorems of $g$-natural metrics on the tangent bundle of a Riemannian manifold $(M, g)$

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Abstract. In [7], it is proved that all $g$-natural metrics on tangent bundles of $m$-dimensional Riemannian manifolds depend on arbitrary smooth functions on positive real numbers, whose number depends on $m$ and on the assumption that the base manifold is oriented, or non-oriented, respectively. The result was originally stated in [8] for the oriented case, but the smoothness was assumed and not explicitly proved. In this note, we shall prove that, both in the oriented and non-oriented cases, the functions generating the $g$-natural metrics are, in fact, smooth on the set of all nonnegative real numbers.

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If $(M, g)$ is an $m$-dimensional Riemannian manifold, then we use the terminology of “$g$-natural metrics” (cf. [2]) on the tangent bundle $TM$ to describe metrics on $TM$ which come from $g$ by a first order natural operator ([8] and [7]). We have studied these metrics in [1], [2] and [3]. The well-known example of such metrics is the Sasaki metric $g^s$ [11]. All natural metrics are characterized by the following result:

**Theorem 1** ([8]). There is a bijective correspondence between natural (possibly degenerated) metrics $G$ on the tangent bundles of (oriented) Riemannian manifolds and the triples of first order natural $F$-metrics $(\zeta_1, \zeta_2, \zeta_3)$, where $\zeta_1$ and $\zeta_3$ are symmetric. The correspondence is given by

$$G = \zeta_1^s + \zeta_2^h + \zeta_3^v,$$

where $\zeta^s$, $\zeta^h$ and $\zeta^v$ denote the Sasaki lift, the horizontal lift and the vertical lift of $\zeta$, respectively.

For the definitions of $F$-metrics and their lifts, we refer to [8] (see also [7] for more details on the concept of naturality).

It is proved, furthermore, in [7] that all first order natural $F$-metrics on (oriented) Riemannian manifolds form a family parameterized by some arbitrary smooth function on positive real numbers, where the number of functions depends on the dimensions of manifolds (the result was originally stated in [8] for the oriented case, but the smoothness was assumed and not explicitly proved). Precisely, with the notations of [7], we have
**Theorem 2** ([7]). 1) All first order natural $F$-metrics $\zeta$ on non-oriented Riemannian manifolds of dimension $m > 1$ form a family parametrized by two arbitrary smooth functions $\alpha, \beta : (0, \infty) \to \mathbb{R}$ in the following way: For every Riemannian manifold $(M, g)$ and tangent vectors $u, X, Y \in M$

\[(1)\quad \zeta(M, g)(u)(X, Y) = \alpha(g(u, u))g(X, Y) + \beta(g(u, u))g(u, X)g(u, Y).\]

If $m = 1$, then the same assertion holds, but we can always choose $\beta = 0$. In particular, all first order natural $F$-metrics are symmetric.

2) On oriented Riemannian manifolds, we have the same results for dimensions $m = 1$ and $m > 3$, but for $m = 2$ and $m = 3$, there exist other arbitrary smooth functions $\phi, \gamma$ and $\delta : (0, \infty) \to \mathbb{R}$ such that:

\[(2)\quad \zeta(M, g)(u)(X, Y) = \alpha(g(u, u))g(X, Y) + \beta(g(u, u))g(u, X)g(u, Y)\]

\[\varphi(g(u, u))g(u, X \times Y),\]

where $\times$ means the vector cross-product. If $m = 2$, then

\[(3)\quad \zeta(M, g)(u)(X, Y) = \alpha(g(u, u))g(X, Y) + \beta(g(u, u))g(u, X)g(u, Y)\]

\[\gamma(g(u, u))(g(J^g(u), X)g(u, Y) + g(u, X)g(J^g(u), Y))\]

\[\delta(g(u, u))(g(J^g(u), X)g(u, Y) - g(u, X)g(J^g(u), Y)),\]

where $J^g$ is the canonical almost complex structure on $(M, g)$.

Actually, the arbitrary parameterizing functions are smooth on all the set of nonnegative real numbers:

**Theorem 3.** All basic functions from Theorem 2 can be prolonged, in fact, to smooth functions on the set $\mathbb{R}^+$ of all nonnegative real numbers.

**Proof:** Note that we will use the technique from [7] throughout the whole proof.

1) Using the same arguments as in [7], we have to discuss all $O(m)$-equivariant maps $\zeta : \mathbb{R}^m \to \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$. Denote by $g^0 = \sum dx^i \otimes dx^i$ the canonical Euclidean metric, and by $\| \|$ the induced norm. Each vector $v \in \mathbb{R}^m$ can be transformed in $|v| \frac{\partial}{\partial x^i}|_0$ by an element of $O(m)$. Hence $\zeta$ is determined by its values on the one-dimensional subspace spanned by $\frac{\partial}{\partial x^i}|_0$. Moreover, we can also change the orientation of the $i$th axis by an element of $O(m)$, i.e., we have to define $\zeta$ only on $\{t \frac{\partial}{\partial x^i}|_0, t \geq 0\}$.

Let us define a smooth map $\xi : \mathbb{R} \to \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$ by $\xi(t) = \zeta(t \frac{\partial}{\partial x^i}|_0) \in \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$, for all $t \in \mathbb{R}$, and consider the group $K_m$ of all linear orthogonal
On the other hand, if we consider the linear orthogonal transformation $A$ changes the orientation of the first coordinate axis, then the equivariance of $g$ so it is a multiple of $A$ even.

For this, applying (a) of Lemma 4 to $A$ and $\nu$ being independent, if $m > 1$. In dimension 1, all $g$-invariant tensors are of the form $\mu^0 g$, the reals $\mu$ and $\nu$ being independent, if $m > 1$.

Thus, our mapping $\xi$ is defined by

$$\xi(t) = \nu(t) d\frac{t}{a(t)} \otimes dx^1 + \mu(t) \mu^0,$$

for all $t \in \mathbb{R}$, where $\mu$ and $\nu$ are arbitrary smooth functions on $\mathbb{R}$ (and they reduce to one function if $m = 1$).

For $t = 0$, since $\xi$ is $O(m)$-invariant, then the tensor $\xi(0)$ is $O(m)$-invariant and so it is a multiple of $g^0$ (cf. [6, I; p. 277]). It follows, by virtue of (5) that $\nu(0) = 0$.

On the other hand, if we consider the linear orthogonal transformation $A_m$ which changes the orientation of the first coordinate axis, then the equivariance of $\xi$ by $A_m$ implies that for every $t \in \mathbb{R}$, $\mu(-t) = \mu(t)$ and $\nu(-t) = \nu(t)$, i.e., $\mu$ and $\nu$ are even.

Now, given $v = t \frac{d}{dt} |_{0}$, $t > 0$, we can write

$$\zeta(\mathbb{R}^m, g^0)(v)(X, Y) = \xi(|v|)(X, Y) = \mu(|v|) g^0(X, Y) + \nu(|v|) |v|^2 g^0(v, X) g^0(v, Y).$$

To complete the proof, we need the following lemma.

**Lemma 4** ([4]). Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function.

(a) If $f$ is even, then there exists a smooth function $g : \mathbb{R}^+ \to \mathbb{R}$ such that $f(t) = f(0) + t^2 g(t^2)$ for any $t$.

(b) If $f$ is odd, then there exists a smooth function $g : \mathbb{R}^+ \to \mathbb{R}$ such that $f(t) = t \cdot f'(0) + t^2 g(t^2)$ for any $t$.

Let us define the functions $\mu(t)$ and $\nu(t)$ by $\nu(t) = t^1 \nu(\sqrt{t})$ and $\mu(t) = \mu(\sqrt{t})$, for all $t > 0$. The functions $\mu$ and $\nu$ being clearly smooth on the set of positive real numbers, it remains to prove that they prolong to smooth functions on $\mathbb{R}^+$. For this, applying (a) of Lemma 4 to $\mu$ and $\nu$, there exist two smooth functions $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}$, such that $\mu(t) = \mu(0) \beta(t^2)$ and $\nu(t) = \nu(0) \alpha(t^2)$.
(since \( \nu(0) = 0 \), for all \( t \in \mathbb{R} \). We deduce that \( \mu(t) = \mu(\sqrt{t}) = \mu(0) + t \alpha(t) \) and \( \nu(t) = t^{-1} \nu(\sqrt{t}) = \beta(t) \), for all \( t > 0 \). In other words, \( \mu \) and \( \nu \) coincide on \( \mathbb{R}^+ \) with two smooth functions on \( \mathbb{R}^+ \), and the formula (1) of Theorem 2 is extended to \( \mathbb{R}^+ \). Obviously, every such operator is natural and 1) of the Theorem is proved.

2) For the oriented situation, when \( m > 3 \) and \( m = 1 \), the same proof remains valid if we replace \( K_m \) by \( K_m^+ := K_m \cap SO(m) \) and \( A_m \) by the element \( B_m \) of \( SO(m) \) which changes the orientations of the first and the second axes.

It remains to extend the formulas (2) and (3) from Theorem 2 to \( \mathbb{R}^+ \). We can use a similar procedure as before.

For \( m = 3 \), let us assume \( s_{ij} dx^i \otimes dx^j \) is \( K_3^+ \)-invariant. If we change the orientation of any coordinate axis, different from the first one, by an element of \( K_3^+ \), then we must change the orientation of the other. It follows that \( s_{12} = s_{21} = s_{13} = s_{31} = 0 \). Further the element of \( K_3^+ \) which exchanges the couple of second and third coordinate axes must change the orientation of one of them, and so \( s_{22} = s_{33} \) and \( s_{23} = -s_{32} \). Hence all \( K_3^+ \)-invariant tensors are of the form

\[
\nu dx^1 \otimes dx^1 + \mu dx^2 \otimes dx^2 + \kappa(dx^2 \otimes dx^3 - dx^3 \otimes dx^2),
\]

the reals \( \mu, \nu \) and \( \kappa \) being independent. Thus, our mapping \( \xi \) is defined by

\[
\xi(t) = \nu(t) dx^1 \otimes dx^1 + \mu(t) dx^2 \otimes dx^2 + \kappa(t)(dx^2 \otimes dx^3 - dx^3 \otimes dx^2),
\]

for all \( t \in \mathbb{R} \), where \( \mu, \nu \) and \( \kappa \) are arbitrary smooth functions on \( \mathbb{R} \). By similar arguments as in 1) we have \( \nu(0) = \kappa(0) = 0 \) and also, if we consider the equivariance of \( \xi \) by \( B_3 \), then we deduce that the functions \( \nu \) and \( \kappa \) are even and that the function \( \kappa \) is odd.

As in 1), let us define \( \mu(t), \nu(t) \) and \( \kappa(t) \) by \( \mu(t) = \mu(\sqrt{t}) \), \( \nu(t) = t^{-1} \nu(\sqrt{t}) \) and \( \kappa(t) = t^{-1/2} \kappa(\sqrt{t}) \) for all \( t > 0 \). The functions \( \mu, \nu \) and \( \kappa \) being clearly smooth on the set of positive real numbers, it remains to prove that they prolong to smooth functions on \( \mathbb{R}^+ \). But we can just apply (a) of Lemma 4 to \( \mu \) and \( \nu \) and (b) of Lemma 4 to \( \kappa \), and the result follows.

For \( m = 2 \), we have \( K_2^+ := K_2 \cap SO(2) = \{ I_2, -I_2 \} \), where \( I_2 \) denotes the identity matrix in \( GL(2) \). Since every tensor in \( \mathbb{R}^{m \times m} \otimes \mathbb{R}^{m \times m} \) is \( K_2^+ \)-invariant, all \( K_2^+ \)-invariant tensors are of the form

\[
\nu dx^1 \otimes dx^1 + \mu dx^2 \otimes dx^2 + \lambda(dx^1 \otimes dx^2 + dx^2 \otimes dx^1) + \tau(dx^2 \otimes dx^2 - dx^2 \otimes dx^1),
\]

the reals \( \mu, \lambda, \tau \) and \( \tau \) being independent. Thus, our mapping \( \xi \) is defined by

\[
\xi(t) = \nu(t) dx^1 \otimes dx^1 + \mu(t) dx^2 \otimes dx^2 + \tau(t)(dx^2 \otimes dx^1 + dx^1 \otimes dx^2)
+ \lambda(t)(dx^2 \otimes dx^1 - dx^1 \otimes dx^2),
\]
for all \( t \in \mathbb{R} \), where \( \mu, \lambda \) and \( \tau \) are arbitrary smooth functions on \( \mathbb{R} \). By similar arguments as in 1) we have \( \nu(0) \lambda(0) = \tau(0) = 0 \) and also all the functions \( \mu, \nu \lambda \) and \( \tau \) being clearly smooth on the set of positive real numbers, it remains to prove that arguments as in 1) we have

\[
\nu(t) = \mu(t) \lambda(t) \tau(t) = \mu(t) \lambda(t) t^{-1} \nu(t)
\]

As in 1), let us define \( \mu(t), \nu(t), \lambda(t) \) and \( \tau(t) \) by \( \mu(t) = \mu(t), \nu(t) = t^{-1} \nu(t), \lambda(t) = t^{-1} \lambda(t) \) and \( \tau(t) = t^{-1} \tau(t) \) for all \( t > 0 \). The functions \( \mu, \nu, \lambda \) and \( \tau \) being clearly smooth on the set of positive real numbers, it remains to prove that they prolong to smooth functions on \( \mathbb{R}^+ \). But we can just apply (a) of Lemma 4 to the functions \( \mu, \lambda \) and \( \tau \) and the result follows.

Combining Theorems 1–3, we obtain for the non-oriented case (an analogous result can be stated for the oriented case):

**Corollary 5.** Let \((M, g)\) be a non-oriented Riemannian manifold and \( G \) be a \( g \)-natural metric on \( TM \). Then there are smooth functions \( \alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}, \)

\[
\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 : \mathbb{R}^+ \to \mathbb{R},
\]

such that for every \( u, X, Y \in TM \), we have

\[
G_{(X, u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2)g(X, Y)
\]

\[(10)\]

\[
+ (\beta_1 + \beta_3)(r^2)g(x, u)_g(Y, u),
\]

\[
G_{(X, u)}(X^h, Y^v) = \alpha_2(r^2)g(X, Y) + \beta_2(r^2)g(x, u)_g(Y, u),
\]

\[
G_{(X, u)}(X^v, Y^h) = \alpha_2(r^2)g(X, Y) + \beta_2(r^2)g(x, u)_g(Y, u),
\]

\[
G_{(X, u)}(X^v, Y^v) = \alpha_1(r^2)g(X, Y) + \beta_1(r^2)g(x, u)_g(Y, u),
\]

where \( r^2 = g(x, u) \).

For \( m = 1 \), the same holds with \( \beta_i = 0, i = 1, 2, 3 \).

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**References**


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