Duality theory of spaces of vector-valued continuous functions

Marian Nowak, Aleksandra Rzepka

Abstract. Let $X$ be a completely regular Hausdorff space, $E$ a real normed space, and let $C_b(X, E)$ be the space of all bounded continuous $E$-valued functions on $X$. We develop the general duality theory of the space $C_b(X, E)$ endowed with locally solid topologies; in particular with the strict topologies $\beta_z(X, E)$ for $z = \sigma, \tau, t$. As an application, we consider criteria for relative weak-star compactness in the spaces of vector measures $M_z(X, E')$ for $z = \sigma, \tau, t$. It is shown that if a subset $H$ of $M_z(X, E')$ is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact, then the set $\text{conv}(S(H))$ is still relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact ($S(H)$ = the solid hull of $H$ in $M_z(X, E')$). A Mackey-Arens type theorem for locally convex-solid topologies on $C_b(X, E)$ is obtained.

Keywords: vector-valued continuous functions, strict topologies, locally solid topologies, weak-star compactness, vector measures

Classification: 46E10, 46E15, 46E40, 46G10

1. Introduction and preliminaries

Let $X$ be a completely regular Hausdorff space and let $(E, \|\cdot\|_E)$ be a real normed space. Let $B_E$ and $S_E$ stand for the closed unit ball and the unit sphere in $E$, and let $E'$ stand for the topological dual of $(E, \|\cdot\|_E)$. Let $C_b(X, E)$ be the space of all bounded continuous functions $f : X \to E$. We will write $C_b(X)$ instead of $C_b(X, \mathbb{R})$, where $\mathbb{R}$ is the field of real numbers. For a function $f \in C_b(X, E)$ we will write $\|f\|_E(x) = \|f(x)\|_E$ for $x \in X$. Then $\|f\| \in C_b(X)$ and the space $C_b(X, E)$ can be equipped with the norm $\|f\|_\infty = \sup_{x \in X} \|f\|_E(x) = \|\|f\|\|_\infty$, where $\|u\|_\infty = \sup_{x \in X} |u(x)|$ for $u \in C_b(X)$.

It turns out that the notion of solidness in the Riesz space (= vector lattice) $C_b(X)$ can be lifted in a natural way to $C_b(X, E)$ (see [NR]). Recall that a subset $H$ of $C_b(X, E)$ is said to be solid whenever $\|f_1\| \leq \|f_2\|$ (i.e., $\|f_1(x)\|_E \leq \|f_2(x)\|_E$ for all $x \in X$ and $f_1 \in C_b(X, E)$, $f_2 \in H$ imply $f_1 \in H$). A linear topology $\tau$ on $C_b(X, E)$ is said to be locally solid if it has a local base at 0 consisting of solid sets. A linear topology $\tau$ on $C_b(X, E)$ that is at the same time locally convex and locally solid will be called a locally convex-solid topology.

In [NR] we examine the general properties of locally solid topologies on the space $C_b(X, E)$. In particular, we consider the mutual relationship between locally solid topologies on $C_b(X, E)$ and $C_b(X)$. It is well known that the so-called
strict topologies \( \beta_z(X, E) \) on \( C_b(X, E) \) (\( z = t, \tau, \sigma, g, p \)) are locally convex-solid topologies (see [Kh, Theorem 8.1], [KhO, Theorem 6], [KhV, Theorem 5]).

For a linear topological space \( (L, \xi) \), by \( (L, \xi)' \) (or \( L'_\xi \)) we will denote its topological dual. We will write \( C_b(X, E)' \) and \( C_b(X)' \) instead of \( (C_b(X, E), \| \cdot \|_\infty)' \) and \( (C_b(X), \| \cdot \|_\infty)' \) respectively. By \( \sigma(L, M) \) and \( \tau(L, M) \) we will denote the weak topology and the Mackey topology with respect to a dual pair \( (L, M) \). For terminology concerning locally solid Riesz spaces we refer to [AB1], [AB2].

In the present paper, we develop the duality theory of the space \( C_b(X, E) \) endowed with locally solid topologies (in particular, the strict topologies \( \beta_z(X, E) \), where \( z = \sigma, \tau, t \)).

In Section 2 we examine the topological dual of \( C_b(X, E) \) endowed with a locally solid topology \( \tau \). We obtain that \( (C_b(X, E), \tau)' \) is an ideal of \( C_b(X, E)' \). We consider a mutual relationship between topological duals of the spaces \( C_b(X) \) and \( C_b(X, E) \), which allows us to examine in a unified manner continuous linear functionals on \( C_b(X, E) \) by means of continuous linear functionals on \( C_b(X) \).

In Section 3 we consider criteria for relative weak-star compactness in spaces of vector measures \( M_2(X, E') \) for \( z = \sigma, \tau, t \). In particular, we show that if a subset \( H \) of \( M_2(X, E') \) is relatively \( \sigma(M_2(X, E'), C_b(X, E)) \)-compact, then \( \text{conv}(S(H)) \) is still relatively \( \sigma(M_2(X, E'), C_b(X, E)) \)-compact (here \( S(H) \) stand for the solid hull of \( H \) in \( M_2(X, E') \); see Definition 3.1 below).

Section 4 deals with the absolute weak and the absolute Mackey topologies on \( C_b(X, E) \). A Mackey-Arens type theorem for locally convex-solid topologies on \( C_b(X, E) \) is obtained.

Now we recall some properties of locally solid topologies on \( C_b(X, E) \) as set out in [NR]. A seminorm \( \rho \) on \( C_b(X, E) \) is said to be solid whenever \( \rho(f_1) \leq \rho(f_2) \) if \( f_1, f_2 \in C_b(X, E) \) and \( \| f_1 \| \leq \| f_2 \| \).

Note that a solid seminorm on the vector lattice \( C_b(X) \) is usually called a Riesz seminorm (see [AB1]).

**Theorem 1.1** (see [NR, Theorem 2.2]). For a locally convex topology \( \tau \) on \( C_b(X, E) \) the following statements are equivalent:

(i) \( \tau \) is generated by some family of solid seminorms;

(ii) \( \tau \) is a locally convex-solid topology.

From Theorem 1.1 it follows that any locally convex-solid topology \( \tau \) on \( C_b(X, E) \) admits a local base at 0 formed by sets which are simultaneously absolutely convex and solid.

Recall that the algebraic tensor product \( C_b(X) \otimes E \) is the subspace of \( C_b(X, E) \) spanned by the functions of the form \( u \otimes e, (u \otimes e)(x) = u(x)e, \) where \( u \in C_b(X) \) and \( e \in E \).

Now we briefly explain the general relationship between locally convex-solid topologies on \( C_b(X) \) and \( C_b(X, E) \) (see [NR]). Given a Riesz seminorm \( p \) on
$C_b(X)$ let us set

$$\rho^\vee (f) := \rho(\|f\|) \text{ for all } f \in C_b(X, E).$$

It is seen that $\rho^\vee$ is a solid seminorm on $C_b(X, E)$. From now on let $e_0 \in S_E$ be fixed. Given a solid seminorm $\rho$ on $C_b(X, E)$ one can define a Riesz seminorm $\hat{\rho}$ on $C_b(X)$ by:

$$\rho^\wedge (u) := \rho(u \otimes e_0) \text{ for all } u \in C_b(X).$$

One can easily show:

**Lemma 1.2 (see [NR, Lemma 3.1]).** (i) If $\rho$ is a solid seminorm on $C_b(X, E)$, then $(\rho^\wedge)^\vee (f) = \rho(f)$ for all $f \in C_b(X, E)$.

(ii) If $p$ is a Riesz seminorm on $C_b(X)$, then $(p^\vee)^\wedge (u) = p(u)$ for all $u \in C_b(X)$.

Let $\tau$ be a locally convex-solid topology on $C_b(X, E)$ and let $\{\rho_\alpha : \alpha \in A\}$ be a family of solid seminorms on $C_b(X, E)$ that generates $\tau$. By $\tau^\wedge$ we will denote the locally convex-solid topology on $C_b(X)$ generated by the family $\{\rho_\alpha^\wedge : \alpha \in A\}$.

Next, let $\xi$ be a locally convex-solid topology on $C_b(X)$ and let $\{\rho_\alpha : \alpha \in A\}$ be a family of solid seminorms on $C_b(X)$ that generates $\xi$. By $\xi^\wedge$ we will denote the locally convex-solid topology on $C_b(X, E)$ generated by the family $\{\rho_\alpha^\wedge : \alpha \in A\}$.

As an immediate consequence of Lemma 1.2 we have:

**Theorem 1.3 (see [NR, Theorem 3.2]).** For a locally convex-solid topology $\tau$ on $C_b(X, E)$ (resp. $\xi$ on $C_b(X)$) we have:

$$(\tau^\wedge)^\vee = \tau \text{ (resp. } (\xi^\wedge)^\vee = \xi).$$

The strict topologies $\beta_z(X, E)$ on $C_b(X, E)$, where $z = t, \tau, \sigma, g, p$ have been examined in [F], [KhC], [K], [KhO1], [KhO2], [KhV1], [KhV2]. In this paper we will consider the strict topologies $\beta_z(X, E)$, where $z = t, \tau, \sigma$. We will write $\beta_t(X, E)$ instead of $\beta_z(X, E)$.

Now we recall the concept of a strict topology on $C_b(X, E)$. Let $\beta X$ stand for the Stone-Cech compactification of $X$. For $v \in C\|X\|_\beta$, $\overline{v}$ denotes its unique continuous extension to $\beta X$. For a compact subset $Q$ of $\beta X \setminus X$ let $C_Q(X) = \{v \in C_b(X) : \overline{v} \mid Q \equiv 0\}$. Let $\beta_Q(X, E)$ be the locally convex topology on $C_b(X, E)$ determined by the family of solid seminorms $\{\vartheta : v \in C_Q(X)\}$, where $\vartheta_v(f) = \sup_{x \in X} |v(x)||f|(x)$ for $f \in C_b(X, E)$.

Now let $C$ be some family of compact subsets of $\beta X \setminus X$. The strict topology $\beta_C(X, E)$ on $C_b(X, E)$ determined by $C$ is the greatest lower bound (in the class of locally convex topologies) of the topologies $\beta_Q(X, E)$, as $Q$ runs over $C$ (see [NR] for more details). In particular, it is known that $\beta_C(X, E)$ is locally solid (see [NR, Theorem 4.1]).
The strict topologies $\beta_{\tau}(X, E)$ and $\beta_{\sigma}(X, E)$ on $C_b(X, E)$ are obtained by choosing the family $C_\tau$ of all compact subsets of $\beta X \setminus X$ and the family $C_\sigma$ of all zero subsets of $\beta X \setminus X$ as $C$, resp. In view of [NR, Corollary 4.4] for $z = \tau, \sigma$ we have

$$\beta_{\tau}(X)^{\vee} = \beta_{\tau}(X, E) \quad \text{and} \quad \beta_{\tau}(X, E)^{\wedge} = \beta_{\tau}(X).$$

The strict topology $\beta_{\tau}(X, E)$ on $C_b(X, E)$ is generated by the family $\{\varphi_v : v \in C_0(X)\}$, where $C_0(X)$ denotes the space of scalar-valued continuous functions on $X$, vanishing at infinity. It is easy to show that

$$\beta_{\tau}(X)^{\vee} = \beta_{\tau}(X, E) \quad \text{and} \quad \beta_{\tau}(X, E)^{\wedge} = \beta_{\tau}(X).$$

2. Topological dual of $C_b(X, E)$ with locally solid topologies

For a linear functional $\|\|$ on $C_b(X, E)$ let us put

$$\|f\| = \sup \{ |(h)| : h \in C_b(X, E), \|h\| \leq \|f\| \}.$$

The next theorem gives a characterization of the space $C_b(X, E)^\prime$.

**Theorem 2.1.** We have

$$C_b(X, E)^\prime = \{ f \in G_b(X, E)^\#: |(f)| < \infty \text{ for all } f \in C_b(X, E) \},$$

where $C_b(X, E)^\#$ denotes the algebraic dual of $C_b(X, E)$.

**Proof:** Indeed, by the way of contradiction, assume that for some $0 \in C_b(X, E)^\prime$ we have $\|f_0\| = \infty$ for some $f_0 \in C_b(X, E)$. Hence there exists a sequence $(h_n)$ in $C_b(X, E)$ such that $\|h_n\| \leq \|f_0\|$ and $\|h_n\| \geq n$ for all $n \in \mathbb{N}$. Since $\|h_n\| \to 0$, we get $n^{-1} \|h_n\| \to 0$, which is in contradiction with $\|h_n\| \geq n$.

Next, assume by the way of contradiction that there exists a linear functional $0$ on $C_b(X, E)$ such that $|0| \|f\| < \infty$ for all $f \in C_b(X, E)$ and $0 \notin C_b(X, E)^\prime$. Then there exists a sequence $(f_n)$ in $C_b(X, E)$ such that $\|f_n\| = 1$ and $|0| \|f_n\| > n^3$ for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3} \|f_n\| = \infty$, the space $(C_b(X), \|\|)$ is complete, there exists $u_0 \in C_b(X)$ such that $\sum_{n=1}^{\infty} \frac{1}{n^3} \|f_n\| = u_0$. Let $f_0 = u_0 \otimes e_0$ for some $x \in C_0$. Then $\frac{1}{n^3} \|f_n\| \leq \|f_0\| = u_0$. Hence for all $n \in \mathbb{N}$, $n < |0| (f_n/n^2) |\leq |0| (f_n/n^2) \leq |0| (f_0) < \infty$, which is impossible. Thus the proof is complete. \(\square\)

Now we consider the concept of solidity in $C_b(X, E)^\prime$.

**Definition 2.1.** For $1, 2 \in C_b(X, E)^\prime$ we will write $1 \leq 2$ whenever $1 \|f\| \leq 2 \|f\|$ for all $f \in C_b(X, E)$. A subset $A$ of $C_b(X, E)^\prime$ is said to be **solid** whenever $1 \leq 2$ with $1 \in C_b(X, E)^\prime$ and $2 \in A$ implies $1 \in A$. A linear subspace $I$ of $C_b(X, E)^\prime$ will be called an **ideal** whenever $I$ is solid.
Since the intersection of any family of solid subsets of \( C_b(X, E)' \) is solid, every subset \( A \) of \( C_b(X, E)' \) is contained in the smallest (with respect to the inclusion) solid set called the solid hull of \( A \) and denoted by \( S(A) \). Note that
\[ S(A) = \{ \| b \| \in C_b(X, E)' : | \| b \| | \leq | \| f \| | \text{ for some } f \in A \}. \]

**Lemma 2.2.** Let \( f \in C_b(X, E)' \). Then for \( f \in C_b(X, E) \),
\[
| \| f \| | = \sup \{ | \| f \| | : \| f \| \in C_b(X, E)' \}, \quad | | \leq | |.
\]
Moreover, if \( A \) is a subset of \( C_b(X, E)' \) then for \( f \in C_b(X, E) \) we have
\[
\sup \{ | \| f \| | : f \in A \} = \sup \{ | \| f \| | : f \in S(A) \}.
\]

**Proof:** Note that \( | | \) is a seminorm on \( C_b(X, E) \). To see that \( | |(f + f_2) \leq | |(f) + | |(f_2) \) holds for \( f_1, f_2 \in C_b(X, E) \) with \( f_1, f_2 \neq 0 \), assume that \( h \in C_b(X, E) \) and \( \| h \| \leq \| f_1 + f_2 \| \). Then for \( h_i = (\| f_i \| / (\| f_1 + f_2 \| ))h \) for \( i = 1, 2 \) we have \( h = h_1 + h_2 \) and \( \| h_i \| \leq \| f_i \| \) for \( i = 1, 2 \). Thus \( | |(h) | \| | + | |(h_2) | \| | \leq | |(f) | \| | + | |(f_2) | \| | \). Hence \( | |(f + f_2) \leq | |(f) + | |(f_2) \), as desired. Moreover, one can easily show that \( | |(\lambda f) | \| | = \lambda | |(f) | \| | \) for all \( \lambda \in \mathbb{R} \).

For a fixed \( f \in C_b(X, E) \) we define a functional \( f \) on the linear subspace \( L_f = \{ \lambda f_0 : \lambda \in \mathbb{R} \} \) of \( C_b(X, E) \) by putting \( f(\lambda f_0) = \lambda | |(f) | \| | \) for \( \lambda \in \mathbb{R} \). It is clear that \( f \) is a linear functional on \( L_f \) and \( f(\lambda f_0) = | |(\lambda f) | \| | \) for \( \lambda \in \mathbb{R} \). Then by the Hahn-Banach extension theorem there exists a linear functional on \( C_b(X, E) \) such that \( | |(f) \) for all \( f \in C_b(X, E) \) and \( f(\lambda f_0) = f(\lambda f_0) \) for all \( \lambda \in \mathbb{R} \). Since \( f \) is linear and \( | |(f) \) and \( | |(-f) \) we get \( | |(f) \) for all \( f \in C_b(X, E) \). To see that \( | |(f) \) we let \( f \in C_b(X, E) \) and take \( h \in C_b(X, E) \) with \( \| h \| \leq \| f \| \). Then \( | |(h) | \| | \leq | |(h) | \| | \) and \( | |(f) \) so \( | |(f) \) and \( | |(f) \) for all \( f \in C_b(X, E) \) under its natural duality
\[ \langle f, \, \rangle = (f) \text{ for } f \in C_b(X, E), \quad \in I \]
will be referred to as a solid dual system.

We now introduce the concept of a solid dual system. Let \( I \) be an ideal of \( C_b(X, E)' \) separating the points of \( C_b(X, E) \). Then the pair \( \langle C_b(X, E), I \rangle \), under its natural duality
\[ \langle f, \, \rangle = (f) \text{ for } f \in C_b(X, E), \quad \in I \]
will be referred to as a solid dual system.

For a subset \( A \) of \( C_b(X, E) \) and a subset \( B \) of \( I \) let us set
\[
A^0 = \{ \in I : | \| f \| | \leq 1 \text{ for all } f \in A \},
\]
\[
B^0 = \{ f \in C_b(X, E) : | \| f \| | \leq 1 \text{ for all } \in B \}.
\]
By making use of Lemma 2.2 we can get the following result.
Theorem 2.3. Let \( \langle C_b(X, E), I \rangle \) be a solid dual system.

(i) If a subset \( A \) of \( C_b(X, E) \) is solid, then \( A^0 \) is a solid subset of \( I \).

(ii) If a subset \( B \) of \( I \) is solid, then \( ^0B \) is a solid subset of \( C_b(X, E) \).

Proof: (i) Let \( |1| \leq |2| \) with \( 1 \in I \) and \( 2 \in A^0 \). Assume that \( f \in A \) and let \( h \in C_b(X, E) \) with \( \|h\| \leq \|f\| \). Then \( h \in A \), because \( A \) is solid, so \( 2(h) \leq 1 \). Hence \( 2(f) \leq 1 \). Thus \( |1(f)| \leq |2(f)| \leq 1 \), so \( 1 \in A^0 \). This means that \( A^0 \) is a solid subset of \( I \).

(ii) Let \( \|f_1\| \leq \|f_2\| \) with \( f_1 \in C_b(X, E) \) and \( f_2 \in ^0B \). To see that \( f_1 \in ^0B \) assume that \( f_1 \in B \). Since \( B \) is a solid subset of \( I \), by Lemma 2.2 the identity \( |(f)| = \sup \{ |(f)| : \alpha \in A \} | \leq |(f)| \) holds. Thus for every \( \alpha \in B \) with \( |(f)| \leq |(f)| \) we have \( |(f)| \leq |(f)| \) and \( |(f)| \leq |(f)| \). Since \( |(f)| \leq |(f)| \), we get \( f_1 \in ^0B \), as desired.

Theorem 2.4. Let \( \tau \) be a locally solid topology on \( C_b(X, E) \). Then \( (C_b(X, E), \tau)' \) is an ideal of \( C_b(X, E)' \).

Proof: To show that \( (C_b(X, E), \tau)' \subset C_b(X, E)' \), by the way of contradiction assume that for some \( 0 \in (C_b(X, E), \tau)' \) we have \( 0 \notin C_b(X, E)' \), so in view of Theorem 2.1 we get \( |0| = \infty \) for some \( f_0 \in C_b(X, E) \). Hence there exists a sequence \( (h_n) \) in \( C_b(X, E) \) such that \( \|h_n\| \leq \|f_0\| \) and \( |0(h_n)| \geq n \) for \( n \in \mathbb{N} \). Since \( n^{-1}f_0 \rightarrow 0 \) for \( \tau \), and \( \tau \) is locally solid, we get \( n^{-1}h_n \rightarrow 0 \) for \( \tau \). Hence \( 0(n^{-1}h_n) \rightarrow 0 \), which is in contradiction with \( |0(h_n)| \geq n \).

To see that \( (C_b(X, E), \tau)' \) is an ideal of \( C_b(X, E)' \), assume that \( |1| \leq |2| \) with \( 1 \in C_b(X, E)' \) and \( 2 \in (C_b(X, E), \tau)' \). Let \( f_\alpha \underset{\tau}{\rightarrow} 0 \) and \( \varepsilon > 0 \) be given. Then there exists a net \( (h_\alpha) \) in \( C_b(X, E) \) such that \( \|h_\alpha\| \leq \|f_\alpha\| \) for each \( \alpha \) and \( 2(|f_\alpha|) \leq |2(h_\alpha)| + \varepsilon \). Clearly \( h_\alpha \underset{\tau}{\rightarrow} 0 \), because \( \tau \) is locally solid, so \( 2(h_\alpha) \rightarrow 0 \). Since \( |1(f_\alpha)| \leq |1(f_\alpha)| \leq |2(f_\alpha)| + \varepsilon \), we get \( 1(f_\alpha) \rightarrow 0 \), so \( 1 \in (C_b(X, E), \tau)' \), as desired.

Theorem 2.5. For a Hausdorff locally convex topology \( \tau \) on \( C_b(X, E) \) the following statements are equivalent:

(i) \( \tau \) is locally solid;

(ii) \( (C_b(X, E), \tau)' \) is an ideal of \( C_b(X, E)' \) and for every \( \tau \)-equicontinuous subset \( A \) of \( (C_b(X, E), \tau)' \) its solid hull \( S(A) \) is also \( \tau \)-equicontinuous.

Proof: (i) \( \Rightarrow \) (ii) By Theorem 2.4 \( (C_b(X, E), \tau)' \) is an ideal of \( C_b(X, E)' \), and thus we have the solid dual system \( (C_b(X, E), (C_b(X, E), \tau)' \). Assume that a subset \( A \) of \( (C_b(X, E), \tau)' \) is equicontinuous. Hence \( A \subset V^0 \) for some solid \( \tau \)-neighbourhood \( V \) of zero. Hence \( S(A) \subset S(V^0) = V^0 \) (see Theorem 2.3). This means that \( S(A) \) is a \( \tau \)-equicontinuous subset of \( (C_b(X, E), \tau)' \).

(ii) \( \Rightarrow \) (i) Let \( B_\tau \) be a local base at zero for \( \tau \) consisting of absolutely convex, \( \tau \)-closed sets. Assume that \( V \) is \( \tau \)-neighbourhood of zero. Then there exists \( U \in B_\tau \),
such that $U \subset V$. Moreover, the polar set $U^0$ is a $\tau$-equicontinuous subset of $(C_b(X, E), \tau)'$. By our assumption $S(U^0)$ is also $\tau$-equicontinuous. Hence there exists $W \in \mathcal{B}_\tau$ such that $W \subseteq S(U^0)$. Since the set $0S(U^0)$ is solid in $C_b(X, E)$, $S(W) \cap 0S(U^0) \subseteq 0(U^0) = \text{abs conv } U^\tau = U \subset V$. This shows that $\tau$ is locally solid, as desired. □

For each $u \in G_b(X, E)'$ let

$$\varphi(u) = \sup \{ \|h\| : h \in C_b(X, E), \|h\| \leq u \} \quad \text{for } u \in C_b(X)^+. $$

One can easily show that $\varphi : C_b(X)^+ \to \mathbb{R}^+$ is an additive and positively homogeneous mapping (see [KhO1, Lemma 1]), so $\varphi$ has a unique positive extension to a linear mapping from $C_b(X)$ to $\mathbb{R}$ (denoted by $\varphi$ again) and given by

$$\varphi(u) = \varphi(u^+) - \varphi(u^-) \quad \text{for all } u \in C_b(X)$$

(see [AB, Lemma 3.1]). Hence $\varphi = |\varphi|$ holds on $C_b(X)^+$. Since $C_b(X)' = C_b(X)^\sim$ (the order dual of $C_b(X)$) (see [AB2, Corollary 12.5]), we get $\varphi \in C_b(X)'$. Moreover, we have:

$$\varphi(||f||) = ||f|| \quad \text{for } f \in G_b(X, E)$$

and

$$\varphi(u) = |(u \otimes \Phi) \quad \text{for } u \in C_b(X)^+. $$

The following lemma will be useful.

**Lemma 2.6.** (i) Assume that $L$ is an ideal of $C_b(X)'$. Then the set

$$C_b(X, E)'_L := \{ \varphi \in G_b(X, E)' : \varphi \in L \}$$

is an ideal of $C_b(X, E)'$.

(ii) Assume that $I$ is an ideal of $C_b(X, E)'$. Then the set

$$C_b(X)'_I := \{ \varphi \in C_b(X)' : |\varphi| \leq \varphi \Phi \quad \text{for some } \Phi \in I \}$$

is an ideal of $C_b(X)'$ and $C_b(X, E)'_{C_b(X)'_I} = I$.

**Proof:** (i) We first show that $G_b(X, E)'_L$ is a linear subspace of $C_b(X, E)'$. Assume that $1, 2 \in C_b(X, E)'_L$, i.e., $\varphi_1, \varphi_2 \in L$. It is easy to show that $\varphi_1 + \varphi_2 (u) \leq (\varphi_1 + \varphi_2) (u)$ for $u \in C_b(X)^+$, so $\varphi_1 + \varphi_2 \in L$, i.e., $1 + 2 \in C_b(X, E)'_L$. Next, let $\varphi \in G_b(X, E)'_L$ and $\lambda \in \mathbb{R}$. Then $\varphi \in L$ and since $\varphi \lambda \Phi = \lambda \varphi \Phi$, we get $\lambda \in G_b(X, E)'_L$. 


To show that $C_b(X,E)'_L$ is solid in $C_b(X,E)'$, assume that $|1| \leq |2|$ with $1 \in C_b(X,E)'$ and $2 \in C_b(X,E)'_L$, i.e., $\varphi_{\Phi_2} \in L$. Then for $u \in C_b(X)^+$ we have $\varphi_{\Phi_1}(u) = |1|(u \circ e_0) \leq |2|(u \circ e_0) = \varphi_{\Phi_2}(u)$. Hence $\varphi_{\Phi_1} \in L$, because $L$ is an ideal of $C_b(X)'$. Thus $1 \in C_b(X,E)'_L$, as desired.

(ii) To prove that $C_b(X)'_L$ is an ideal of $C_b(X)'$, assume that $|\varphi_1| \leq |\varphi_2|$, where $\varphi_1, \varphi_2 \in C_b(X)'$. Then $|\varphi_2| \leq \varphi_\Phi$ for some $\Phi \in I$, so $|\varphi_1| \leq \varphi_\Phi$, and this means that $\varphi_1 \in C_b(X)'_L$.

To show that $I \subset C_b(X,E)'_L$, assume that $\varphi \in C_b(X)'_L$, so $\varphi \in G_b(X,E)'_{C_b(X)'_L}$. Now, we assume that $\varphi \in G_b(X,E)'_{C_b(X)'_L}$, i.e., $\varphi \in C_b(X,E)'$ and $\varphi_\Phi \in C_b(X)'_L$. It follows that there exists $0 \in I$ such that $\varphi_\Phi \leq \varphi_\Phi_0$. Hence for every $f \in C_b(X,E)$ we have $|f| = \varphi_\Phi(\|f\|) \leq \varphi_\Phi_0(\|f\|) = |f|$. Thus $f \in I$, because $I$ is an ideal of $C_b(X,E)'_L$.

Let $A$ be a subset of $C_b(X,E)'_L$. Then $S(A) \subset C_b(X,E)'_L$ as $C_b(X,E)'_L$ is solid (by Theorem 2.4). Hence

$$S(A) = \{ \varphi \in G_b(X,E)'_L : |\varphi| \leq |f| \} \text{ for some } f \in A.$$  

In view of Lemma 2.2 for a subset $A$ of $C_b(X,E)'$ and $f \in C_b(X,E)$ we have:

$$\sup \{ |f| : \varphi \in A \} = \sup \{ |\varphi(\|f\|) : \varphi \in A \} = \sup \{ |f| : \varphi \in S(A) \}.$$  

**Theorem 2.7.** Let $\tau$ be a locally convex-solid Hausdorff topology on $G(X,E)$. Then for a subset $A$ of $G_b(X,E)'$ the following statements are equivalent:

(i) $A$ is $\tau$-equicontinuous;

(ii) $\text{conv}(S(A))$ is $\tau$-equicontinuous;

(iii) $S(A)$ is $\tau$-equicontinuous;

(iv) the subset $\{\varphi_\Phi : \Phi \in A\}$ of $G_b(X)'$ is $\tau^\wedge$-equicontinuous.

**Proof:** (i) $\implies$ (ii) In view of Theorem 2.4 we have a solid dual system $(C_b(X,E), C_b(X,E)'_L)$. Let $A$ be $\tau$-equicontinuous. Then by Theorem 1.1 there is a convex solid $\tau$-neighbourhood $V$ of zero such that $A \subset V^0$. Hence $\text{conv}(S(A)) = \text{conv}(S(V^0)) = V^0$ (see Theorem 2.3), and this means that $\text{conv}(S(A))$ is still $\tau$-equicontinuous.

(ii) $\implies$ (iii) It is obvious.

(iii) $\implies$ (iv) Assume that the subset $S(A)$ of $C_b(X,E)'$ is $\tau$-equicontinuous. Let $\{\rho_\alpha : \alpha \in A\}$ be a family of solid seminorms on $C_b(X,E)$ that generates $\tau$. Given $\varepsilon > 0$ there exist $\alpha_1, \ldots, \alpha_n \in A$ and $\eta > 0$ such that $\sup \{|\varphi_\Phi(\|f\|) : \varphi \in S(A)\} \leq \varepsilon$. 


whenever \( \rho_{u_i}(f) \leq \eta \) for \( i = 1, 2, \ldots, n \). To show that \( \{ \varphi_\phi : \phi \in A \} \) is \( \tau \)-equicontinuous, it is enough to show that \( \sup \{ \varphi_\phi(u) : \phi \in \rho \} \leq \varepsilon \) whenever \( \rho_{u_i}(u) \leq \eta \) for \( i = 1, 2, \ldots, n \). Indeed, let \( u \in C_b(X) \) and \( \rho_{u_i}(u) \leq \eta \) for \( i = 1, 2, \ldots, n \). Then \( \rho_{u_i}(u \otimes e_0) \leq \eta \) \( (i = 1, 2, \ldots, n) \), so \( \sup \{ \| u \otimes e_0 \| : \phi \in \rho \} \leq \varepsilon \). Hence, in view of \( (+) \) we obtain that \( \sup \{ \varphi_\phi(u) : u \in A \} \leq \varepsilon \), because \( \| u \otimes e_0 \| = |u| \). But \( |\varphi_\phi(u)| \leq \varphi_\phi(|u|) \), and the proof is complete.

(iv) \( \rightarrow \) (i) Assume that the set \( \{ \varphi_\phi : \phi \in A \} \) is \( \tau \)-equicontinuous. Let \( \{ \rho_\alpha : \alpha \in A \} \) be a family of solid seminorms on \( C_b(X, E) \) that generates \( \tau \). Given \( \varepsilon > 0 \) there exist \( \alpha_1, \ldots, \alpha_n \in A \) and \( \eta > 0 \) such that \( \sup \{ \| \varphi_\phi(u) \| : \phi \in \rho \} \leq \varepsilon \) whenever \( u \in C_b(X) \) and \( \rho_{u_i}(u) \leq \eta \) for \( i = 1, 2, \ldots, n \). Let \( f \in C_b(X, E) \) with \( \rho_\alpha(f) \leq \eta \) for \( i = 1, 2, \ldots, n \). Since \( \rho_\alpha(\| f \|) = \rho_\alpha(\| f \| \otimes e_0) = \rho_{u_i}(f) \) \( (i = 1, 2, \ldots, n) \), \( \sup \{ \| \varphi_\phi(\| f \|) \| : \phi \in A \} \leq \varepsilon \). But \( \| f \| \leq \| f \phi(\| f \|) \| \), so \( \sup \{ \| (f) \| : \phi \in A \} \leq \varepsilon \). This means that \( A \) is \( \tau \)-equicontinuous. \( \square \)

**Corollary 2.8.** Let \( \tau \) be a locally convex-solid topology on \( C_b(X, E) \). Then for \( \in C_b(X, E)' \) the following statements are equivalent:

(i) \( \varphi_\phi \) is \( \tau \)-continuous;

(ii) \( \varphi_\phi \) is \( \tau^\wedge \)-continuous.

**Corollary 2.9.** Let \( \xi \) be a locally convex-solid topology on \( C_b(X) \). Then for \( \in C_b(X, E)' \) the following statements are equivalent:

(i) \( \varphi_\phi \) is \( \xi \)-continuous;

(ii) \( \varphi_\phi \) is \( \xi^\wedge \)-continuous.

**Remark.** For the equivalence (i) \( \iff \) (iv) of Theorem 2.7 for the strict topologies \( \beta(X, E) \) \( (z = \sigma, \tau, t, \infty, g) \) see [KhO_3, Lemma 2];

**Corollary 2.10.** (i) Let \( \xi \) be a locally convex-solid topology on \( C_b(X) \). Then

\[
(C_b(X), \xi)' = \left\{ \varphi \in C_b(X)': |\varphi| \leq \varphi_\phi \text{ for some } \varphi \in (C_b(X, E), \xi')' \right\}.
\]

(ii) Let \( \tau \) be a locally convex-solid topology on \( C_b(X, E) \). Then

\[
(C_b(X, E), \tau)' = \left\{ \varphi \in C_b(X)': |\varphi| \leq \varphi_\phi \text{ for some } \varphi \in (C_b(X, E), \tau')' \right\}.
\]

**Proof:** (i) Let \( \varphi \in (C_b(X), \xi)' \). Define a linear functional \( \phi \) on the subspace \( C_b(X)(e_0) = \{ u \otimes e_0 : u \in C_b(X) \} \) of \( C_b(X, E) \) by putting \( \phi(u \otimes e_0) = \varphi(u) \) for \( u \in C_b(X) \). Let \( \{ \rho_\alpha : \alpha \in A \} \) be a family of Riesz seminorms generating \( \xi \). Since \( \varphi \in (C_b(X), \xi)' \), there exist \( \varepsilon > 0 \) and \( \alpha_1, \ldots, \alpha_n \in A \) such that for \( u \in C_b(X) \)

\[
| \phi(u \otimes e_0) | = |\varphi(u)| \leq c \max_{1 \leq i \leq n} \rho_{\alpha_i}(u) = c \max_{1 \leq i \leq n} \rho_{\alpha_i}(u \otimes e_0).
\]
This means that \( 0 \in (\mathcal{C}_b(X)(e_0), \xi' \mid \mathcal{C}_b(X)(e_0))' \), so by the Hahn-Banach extension theorem there is \( u \in (\mathcal{C}_b(X,E), \xi')' = \mathcal{C}_b(X,E) \) such that \( u \otimes e_0 = \varphi(u) \) for all \( u \in \mathcal{C}_b(X) \). We shall now show that \( |\varphi| \leq \varphi_{\mathcal{F}} \), i.e., \( |\varphi|(u) \leq \varphi_{\mathcal{F}}(u) \) for all \( u \in \mathcal{C}_b(X)^+ \). Indeed, let \( u \in \mathcal{C}_b(X)^+ \) be given and let \( v \in \mathcal{C}_b(X) \) with \( |v| \leq u \). Then we have \( |\varphi(v)| = |(v \otimes e_0)| \leq \varphi_{\mathcal{F}}(u) \), so \( |\varphi| \leq \varphi_{\mathcal{F}} \), as desired.

Next, assume that \( \varphi \in \mathcal{C}_b(X)' \) with \( |\varphi| \leq \varphi_{\mathcal{F}} \) for some \( \varphi_{\mathcal{F}} \in (\mathcal{C}_b(X), \xi)' \). In view of Corollary 2.9, \( \varphi_{\mathcal{F}} \in (\mathcal{C}_b(X), \xi)' \) and since \( (\mathcal{C}_b(X), \xi)' \) is an ideal of \( \mathcal{C}_b(X)' \), we conclude that \( \varphi \in (\mathcal{C}_b(X), \xi)' \).

(ii) It follows from (i), because \( (\tau^\wedge)^\vee = \tau \). \( \square \)

It is well known that if \( L \) is a \( \sigma \)-Dedekind complete vector-lattice and if \( H \) is a relatively \( \sigma(L_{\mathcal{N}}, L) \)-compact subset of \( L_{\mathcal{N}} \) (resp. a relatively \( \sigma(L_{\mathcal{C}}, L) \)-compact subset of \( L_{\mathcal{C}} \)), then the set \( \operatorname{conv}(S(H)) \) is still relatively \( \sigma(L_{\mathcal{N}}, L) \)-compact (resp. relatively \( \sigma(L_{\mathcal{C}}, L) \)-compact) (see [AB, Corollary 20.12, Corollary 20.10]) (here \( L_{\mathcal{N}} \) and \( L_{\mathcal{C}} \) stand for the order continuous dual and the \( \sigma \)-order continuous dual of \( L \) resp.).

Now, we shall show that this property holds in \( (\mathcal{C}_b(X,E)'_{\beta_z}, \sigma(\mathcal{C}_b(X,E)'_{\beta_z}, \mathcal{C}_b(X,E))) \) for \( z = \sigma, \tau, t \).

Recall that a completely regular Hausdorff space \( X \) is called a \( P \)-space if every \( G_\delta \) set in \( X \) is open (see [GJ, p. 63]).

The following result will be of importance.

**Theorem 2.11.** Let \( H \) be a norm-bounded and \( \sigma(\mathcal{C}_b(X,E)'_{\beta_z}, \mathcal{C}_b(X,E)) \)-compact subset of \( \mathcal{C}_b(X,E)'_{\beta_z} \), where \( z = \sigma \) (resp. \( z = \tau \) and \( X \) is a paracompact space; resp. \( z = \tau \) and \( X \) is a \( P \)-space). Then \( H \) is \( \beta_z(X,E) \)-equicontinuous.

**Proof:** See [KhO, Theorem 5] for \( z = \sigma \); [Kh, Theorem 6.1] for \( z = \tau \) and [KhC, Lemma 3] for \( z = t \). \( \square \)

Now we are ready to state our main result.

**Theorem 2.12.** Let \( H \) be a norm bounded subset of \( \mathcal{C}_b(X,E)'_{\beta_z} \), where \( z = \sigma \) (resp. \( z = \tau \) and \( X \) is a paracompact space; resp. \( z = t \) and \( X \) is a \( P \)-space). Then the following statements are equivalent:

(i) \( H \) is relatively countably \( \sigma(\mathcal{C}_b(X,E)'_{\beta_z}, \mathcal{C}_b(X,E)) \)-compact;
(ii) \( H \) is \( \beta_z(X,E) \)-equicontinuous;
(iii) \( \operatorname{conv}(S(H)) \) is relatively \( \sigma(\mathcal{C}_b(X,E)'_{\beta_z}, \mathcal{C}_b(X,E)) \)-compact;
(iv) \( S(H) \) is relatively \( \sigma(\mathcal{C}_b(X,E)'_{\beta_z}, \mathcal{C}_b(X,E)) \)-compact;
(v) \( H \) is relatively \( \sigma(\mathcal{C}_b(X,E)'_{\beta_z}, \mathcal{C}_b(X,E)) \)-compact.

**Proof:** (i) \( \implies \) (ii) See Theorem 2.21.

(ii) \( \implies \) (iii) In view of Theorem 2.7 the set \( \operatorname{conv}(S(H)) \) is \( \beta_z(X,E) \)-equicontinuous, i.e., there is a neighbourhood of 0 for \( \beta_z(X,E) \) such that \( \operatorname{conv}(S(H)) \subset V^0 \).
Duality theory of spaces of vector-valued continuous functions

(= the polar set with respect to the dual pair \( \langle C_b(X, E), C_b(X, E)^\prime \rangle \)). Then by the Banach-Alaoglu’s theorem the set \( V^0 \) is \( \sigma(C_b(X, E)^\prime, C_b(X, E)) \)-compact, so the set \( \text{conv}(S(H)) \) is relatively \( \sigma(C_b(X, E)^\prime, C_b(X, E)) \)-compact.

(iii) \( \implies \) (iv) \( \implies \) (v) \( \implies \) (i) It is obvious. \( \square \)

3. Weak-star compactness in some spaces of vector measures

In this section we consider criteria for relative weak-star compactness in some spaces of vector measures \( M_z(X, E^\prime) \) for \( z = \sigma, \tau, t \). In particular, by making use of Theorem 2.11 we show that if a subset \( H \) of \( M_z(X, E^\prime) \) is relatively \( \sigma(M_z(X, E^\prime), C_b(X, E)) \)-compact, then the set \( \text{conv}(S(H)) \) is still relatively \( \sigma(M_z(X, E^\prime), C_b(X, E)) \)-compact (here \( S(H) \) stand for the solid hull of \( H \)). We start by recalling some notions and results concerning the topological measure theory (see [V], [S], [Wh]).

Let \( B(X) \) be the algebra of subsets of \( X \) generated by the zero sets. Let \( M(X) \) be the space of all bounded \( \sigma \)-additive regular (with respect to the zero sets) measures on \( B(X) \). The spaces of all \( \sigma \)-additive, \( \tau \)-additive and tight members of \( M(X) \) will be denoted by \( M_{\sigma}(X) \), \( M_{\tau}(X) \) and \( M_t(X) \) respectively (see [V], [Wh]). It is well known that \( M_z(X) \) for \( z = \sigma, \tau, t \) are ideals of \( M(X) \) (see [Wh, Theorem 7.2]).

Theorem 3.1 (A.D. Alexandro ; [Wh, Theorem 5.1]). For a linear functional \( \varphi : C_b(X) \to \mathbb{R} \) the following statements are equivalent.

(i) \( \varphi \in C_b(X)^\prime \).

(ii) There exists a unique \( \mu \in M(X) \) such that

\[
\varphi(u) = \varphi_\mu(u) = \int_X u \, d\mu \quad \text{for all } \ u \in C_b(X).
\]

Moreover, \( \mu \geq 0 \) if and only if \( \varphi_\mu(u) \geq 0 \) for all \( u \in C_b(X)^+ \).

By \( M(X, E^\prime) \) we denote the set of all \( \sigma \)-additive measures \( m : B(X) \to E \) with the following properties:

(i) For every \( e \in E \), the function \( m_e : B(X) \to \mathbb{R} \) is defined by \( m_e(A) = m(A)(e) \), belongs to \( M(X) \).

(ii) \( |m|(X) < \infty \), where for \( A \in B(X) \)

\[
|m|(A) = \sup \left\{ \sum_{i=1}^n m(B_i)(e_i) : \bigcup_{i=1}^n B_i = A, B_i \in B(X), B_i \cap B_j = \emptyset \text{ for } i \neq j, \ e_i \in E, \ n \in \mathbb{N} \right\}.
\]
For $z = \sigma, \tau, t$ let

$$M_z(X, E') = \{ m \in M(X, E') : m_\varepsilon \in M_z(X) \text{ for every } \varepsilon \in E \}.$$  

It is well known that $|m| \in M(X)$ (resp. $|m| \in M_z(X)$ for $z = \sigma, \tau, t$) whenever $m \in M(X, E')$ (resp. $m \in M_z(X, E')$) for $z = \sigma, \tau, t$ (see [F, Proposition 3.9]).

Now we are ready to define the notion of solidness in $M(X, E)$.

**Definition 3.1.** For $m_1, m_2 \in M(X, E')$ we will write $|m_1| \leq |m_2|$ whenever $|m_1|(|B|) \leq |m_2|(|B|)$ for every $B \in B(X)$. A subset $H$ of $M(X, E')$ is said to be **solid** whenever $|m_1| \leq |m_2|$ with $m_1 \in M(X, E')$ and $m_2 \in H$ implies $m_1 \in H$.

A linear subspace $I$ of $M(X, E')$ will be called an **ideal** of $M(X, E')$ whenever $I$ is a solid subset of $M(X, E')$.

**Proposition 3.2.** $M_z(X, E')$ ($z = \sigma, \tau, t$) is an ideal of $M(X, E')$.

**Proof:** Let $|m_1| \leq |m_2|$, where $m_1 \in M(X, E')$ and $m_2 \in M_z(X, E')$. Then $|m_1| \in M(X)$ and $|m_2| \in M_z(X)$, and since $M_z(X)$ is an ideal of $M(X)$ we conclude that $|m_1| \in M_z(X)$. For each $e \in E$ we have $|(m_1)_e|(B) \leq \|e\|_E |m_1|(B)$ for $B \in B(X)$, so $(m_1)_e \in M_z(X)$, i.e., $m_1 \in M_z(X, E')$.

Since the intersection of any family of solid subsets of $M(X, E')$ is solid, every subset $H$ of $M(X, E')$ is contained in the smallest (with respect to inclusion) solid set called the **solid hull of** $H$ and denoted by $S(H)$. Note that

$$S(H) = \{ m \in M(X, E') : |m| \leq |m'| \text{ for some } m' \in H \}.$$  

Now we recall some results concerning a characterization of the topological duals of $(C_b(X, E), \beta_z(X, E))$ in terms of the spaces $M_z(X, E')$ ($z = \sigma, \tau, t$).

**Theorem 3.3.** Assume that $\beta_z(X, E)$ is the strict topology on $C_b(X, E)$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\tau(X, E))$ (resp. $z = \tau$; resp. $z = t$).

Then for a linear functional $\varphi \in G(X, E)$ the following statements are equivalent.

(i) is $\beta(X, E)$-continuous.

(ii) There exists a unique $m \in M_z(X, E')$ such that

$$f = m(f) = \int_X f \, dm \quad \text{for every } f \in C_b(X, E).$$

(iii) The functional $\varphi_\beta$ is $\beta_z(X)$-continuous.

Moreover, $\|m\| = |m|(X)$ for $m \in M_z(X, E')$.

**Proof:** (i) $\iff$ (ii) See [Kh, Theorem 5.3] for $z = \sigma$; [Kh, Corollary 3.9] for $z = \tau$; [F1, Theorem 3.13] for $z = t$.

(ii) $\iff$ (iii) It follows from Corollary 2.8, because $\beta_z(X, E) = \beta_z(X)$.  

\[ \square \]
Lemma 3.4. Assume that $m \in M_z(X, E')$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma(X, E))$ (resp. $z = \tau$; resp. $z = t$). Then

$$\varphi_{[m]}(u) = \int_X u \, dm = \varphi_{[m]}(u) \quad \text{for all} \quad u \in C_b(X).$$

**Proof:** Let $u \in C_b(X)^+$ and $m \in M_z(X, E')$. Then for $h \in C_b(X, E)$ with $\|h\| \leq u$ by [F2, Lemma 3.11] we have

$$|m(h)| = \left| \int_X h \, dm \right| \leq \int_X \|h\| \, dm \leq \int_X u \, dm = \varphi_{[m]}(u).$$

Hence

$$\varphi_{[m]}(u) = |\, m(u \otimes e_0) = \sup \{|m(h)| : h \in C_b(X, E), \|h\| \leq u\} \leq \varphi_{[m]}(u).$$

On the other hand, in view of [Kh, Theorem 2.1] we have

$$\varphi_{[m]}(u) = \int_X u \, dm = \sup \{|m(g)| : g \in C_b(X) \otimes E, \|g\| \leq u\},$$

so $\varphi_{[m]}(u) \leq \varphi_{[m]}(u)$. Thus $\varphi_{[m]}(u) = \varphi_{[m]}(u)$ for all $u \in C_b(X)$. \hfill $\Box$

Lemma 3.5. Assume that $m_1, m_2 \in M_z(X, E')$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma(X, E))$ (resp. $z = \tau$; resp. $z = t$). Then the following statements are equivalent:

(i) $|m_1| \leq |m_2|$, i.e., $|m_1|(B) \leq |m_2|(B)$ for every $B \in B(X)$;

(ii) $\varphi_{[m_1]}(u) \leq \varphi_{[m_2]}(u)$ for every $u \in C_b(X)^+$;

(iii) $|m_1|(f) \leq |m_2|(f)$ for every $f \in C_b(X, E)$.

**Proof:** (i) $\iff$ (ii) It easily follows from Theorem 3.1.

(ii) $\implies$ (iii) In view of Lemma 3.4 we get

$$|m_1|(f) = \varphi_{[m_1]}(\|f\|) = \varphi_{[m_1]}(\|f\|) \leq \varphi_{[m_2]}(\|f\|) = \varphi_{[m_2]}(\|f\|) = |m_2|(f).$$

(iii) $\implies$ (ii) By Lemma 3.3 for $u \in C_b(X)^+$ and $e_0 \in S_E$ we have

$$\varphi_{[m_1]}(u) = \varphi_{[m_1]}(u) = |m_1|(u \otimes e_0) \leq |m_2|(u \otimes e_0) = \varphi_{[m_2]}(u) = \varphi_{[m_2]}(u).$$

\hfill $\Box$
Lemma 3.6. Assume that $H \subset M_z(X, E')$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E'), \beta_\sigma(X, E))$ (resp. $z = \tau$; resp. $z = t$), and let $H = \{ m : m \in H \}$. Then $\text{conv} (S(H)) = \text{conv} (S(H))$.

Proof: Assume that $\in \text{conv} (S(H))$. Then $= \sum_{i=1}^{n} \alpha_i m_i = \sum_{i=1}^{n} \alpha_i m_i$, where $m_i \in M_z(X, E')$ and $\alpha_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$, and $|m_i| \leq |m'_i|$ for some $m'_i \in H$ and $i = 1, 2, \ldots, n$. In view of Lemma 3.5 $|m_i| \leq |m'_i|$, i.e., $m_i \in S(H)$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i m_i \in \text{conv} (S(H))$. This means that $\in \text{conv} (S(H))$.

Assume that $\in \text{conv} (S(H))$. Then $= \sum_{i=1}^{n} \alpha_i m_i = \sum_{i=1}^{n} \alpha_i m_i$, where $m_i \in M_z(X, E')$ and $\alpha_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$, and $|m_i| \leq |m'_i|$ for some $m'_i \in H$ and $i = 1, 2, \ldots, n$. By Lemma 3.5 $|m_i| \leq |m'_i|$ for $i = 1, 2, \ldots, n$, so $\in \text{conv} (S(H))$.

Corollary 3.7. Assume that $m_0 \in M_z(X, E')$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E'), \beta_\sigma(X, E))$ (resp. $z = \tau$; resp. $z = t$) and let $e \in S_E$. Then for every $u \in C_b(X)^+$ we have:

$$\int_X u \, d|m_0| = \sup \left\{ \left| \int_X u \, dm_e \right| : m \in M_z(X, E'), |m| \leq |m_0| \right\}.$$

Proof: Let $m_0 \in M_z(X, E')$ and $e \in S_E$. Assume that $\in G(X, E)'$ and $| \leq | \leq | m_0 |$. Since $m_0 \in C_b(X, E)'^{\beta_\sigma}$ (see Theorem 3.3), by making use of Theorem 2.4 we get $\in G(X, E)'^{\beta_\sigma}$. Hence in view of Theorem 3.3 and Lemma 3.5 we see that $m$ for some $m \in M_z(X, E')$ with $|m| \leq |m_0|$. Moreover, it is easy to observe that for every $u \in M(X, E')$ and $u \in C_b(X)$ we have:

$$\int_X (u \otimes e) \, dm = \int_X u \, dm_e.$$

Thus in view of Lemma 3.4, Lemma 2.2 and Lemma 3.5 we get:

$$\int_X u \, d|m_0| = \varphi_{m_0}(u) = \left| m_0 \right|(u \otimes e)$$

$$= \sup \left\{ \left| (u \otimes e) \right| : \in G(X, E)', \left| \right| \leq \left| m_0 \right| \right\}$$

$$= \sup \left\{ \left| m(u \otimes e) \right| : m \in M_z(X, E'), \left| m \right| \leq \left| m_0 \right| \right\}$$

$$= \sup \left\{ \left| \int_X (u \otimes e) \, dm \right| : m \in M_z(X, E'), \left| m \right| \leq \left| m_0 \right| \right\}$$

$$= \sup \left\{ \left| \int_X u \, dm_e \right| : m \in M_z(X, E'), \left| m \right| \leq \left| m_0 \right| \right\}.$$

To state our main result we recall some definitions (see [Wh, Definition 11.13, Definition 11.23, Theorem 10.3]).
A subset $A$ of $M_2(X)$ (resp. $M_2(X)$) is said to be uniformly $\sigma$-additive (resp. uniformly $\tau$-additive) if whenever $u_n(x) \downarrow 0$ for every $x \in X$, $u_n \in C_b(X)^+$ (resp. $u_n \downarrow 0$), then $\sup\{\int_X u_n\,d\mu : \mu \in A\} \to 0$ (resp. $\sup\{\int_X u_n\,d\mu : \mu \in A\} \to 0$).

A subset $A$ of $M_1(X)$ is said to be uniformly tight if given $\varepsilon > 0$ there exists a compact subset $K$ of $X$ such that $\sup\{|\mu|(X \setminus K) : \mu \in A\} \leq \varepsilon$.

Now we are in position to prove our desired result.

**Theorem 3.8.** For a subset $H$ of $M_2(X, E')$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma(X, E))$ (resp. $z = \tau$ and $X$ is paracompact; resp. $z = t$ and $X$ is a $P$-space) the following statements are equivalent.

(i) $H$ is relatively $\sigma(M_2(X, E'), C_b(X, E))$-compact.

(ii) $\text{conv}(S(H))$ is relatively $\sigma(M_2(X, E'), C_b(X, E))$-compact.

(iii) The set $\{|\mu| : m \in H\}$ in $M_2(X)^+$ is uniformly $\sigma$-additive for $z = \sigma$, (resp. uniformly $\tau$-additive for $z = \tau$; resp. uniformly tight for $z = t$).

**Proof:** (i) $\implies$ (ii) It is seen that $H$ is relatively $\sigma(M_2(X, E'), C_b(X, E))$-compact if and only if $H$ is relatively $\sigma(C_b(X, E)', \beta_\sigma(X, E))$-compact. Hence by Theorem 2.12 and Lemma 3.6 the set $\text{conv}(S(H))$ is still relatively $\sigma(C_b(X, E)', \beta_\sigma(X, E))$-compact. This means that $\text{conv}(S(H))$ is relatively $\sigma(M_2(X, E'), C_b(X, E))$-compact.

(ii) $\implies$ (i) It is obvious.

(i) $\iff$ (iii) In view of Theorem 2.12 $H$ is relatively $\sigma(M_2(X, E'), C_b(X, E))$-compact if and only if $H$ is $\beta_\sigma(X, E)$-equicontinuous; hence in view of Theorem 2.7 and Lemma 3.4 the subset $\{\phi_{\mu} : m \in H\}$ of $(C_b(X, \beta_\sigma(X))(E))'$ is $\beta_\sigma(X)$-equicontinuous. It is known that the subset $\{\phi_{\mu} : m \in H\}$ of $(C_b(X, \beta_\sigma(X))(E))'$ is $\beta_\sigma(X)$-equicontinuous if and only if the set $\{|\mu| : m \in H\}$ in $M_2(X)^+$ is uniformly $\sigma$-additive for $z = \sigma$ (see [Wh, Theorem 11.14]) (resp. uniformly $\tau$-additive for $z = \tau$ (see [Wh, Theorem 11.24]); resp. uniformly tight for $z = t$ (see [Wh, Theorem 10.7])).

4. **A Mackey-Arens type theorem for locally convex-solid topologies on $C_b(X, E)$**

Let $I$ be an ideal of $C_b(X, E)'$ separating points of $C_b(X, E)$. For each $\phi \in I$ let us put

$$\rho_{\phi}(f) = |\phi(f)| \quad \text{for} \quad f \in C_b(X, E).$$

One can show that $\rho_{\phi}$ is a solid seminorm on $C_b(X, E)$ (see the proof of Lemma 2.2). We define the absolute weak topology $|\sigma|([\phi(X, E), I])$ on $C_b(X, E)$ as

$$|\sigma|([\phi(X, E), I]) = \{f \in C_b(X, E) : \rho_{\phi}(f) \leq 1, \phi \in I\}.$$
the locally convex-solid topology generated by the family \( \{ \rho_{\varphi} : \varphi \in I \} \). In view of Lemma 2.2 we have
\[
\rho_{\varphi}(f) = | |(f) = \sup \{ | \varphi(f) : \varphi \in I, \quad | \varphi | \leq \rho_{\varphi} \}
\]
This means that \(|\sigma|(C_b(X, E), I)\) is the topology of uniform convergence on sets of the form \( \{ f \in I : | \varphi(f) | \leq | \} \} \) = \( S\{ \} \), where \( \in I \).

Assume that \( L \) is an ideal of \( C_b(X)' \) separating the points of \( C_b(X) \). For each \( \varphi \in L \) the function \( p_{\varphi}(u) = | \varphi(u) | \) for \( u \in C_b(X) \) defines a Riesz semi-norm on \( C_b(X) \). The family \( \{ p_{\varphi} : \varphi \in I \} \) defines a locally convex-solid topology \(|\sigma|(C_b(X), L)\) on \( C_b(X) \), called the absolute weak topology generated by \( L \) (see [AB]).

Recall that \(|\sigma|(C_b(X), L)^{\lor}\) is the locally convex-solid topology on \( C_b(X, E) \) generated by the family \( \{ p_{\varphi}^{\lor} : \varphi \in L \} \), where \( p_{\varphi}^{\lor}(f) = p_{\varphi}(\|f\|) \) for \( f \in C_b(X, E) \).

We shall need the following result.

**Lemma 4.1.** Let \( I \) be an ideal of \( C_b(X, E)' \) separating the points of \( C_b(X, E) \). Then
\[
|\sigma|(C_b(X, E), I) = |\sigma|(C_b(X), C_b(X)_I')^{\lor}
\]
where \( C_b(X)_I' = \{ \varphi \in C_b(X)' : | \varphi | \leq \varphi_{\varphi} \text{ for some } \varphi \in I \} \).

**Proof:** Let \( \varphi \in C_b(X)' \), i.e., \( | \varphi | \leq \varphi_{\varphi} \) for some \( \varphi \in I \). Then for \( f \in G_b(X, E) \) we have
\[
p_{\varphi}^{\lor}(f) = p_{\varphi}(\|f\|) = | \varphi(\|f\|) | \leq \varphi_{\varphi}(\|f\|) = | |(f) = \rho_{\varphi}(f).
\]
This means that \(|\sigma|(C_b(X), C_b(X)_I)^{\lor} \subset |\sigma|(C_b(X, E), I)\).

Next, let \( \varphi \in I \). Then for \( f \in G_b(X, E) \) we have
\[
\rho_{\varphi}(f) = | |(f) = \varphi(\|f\|) = p_{\varphi}(\|f\|) = p_{\varphi}^{\lor}(f).
\]
This shows that \(|\sigma|(C_b(X, E), I) \subset |\sigma|(C_b(X), C_b(X)_I)^{\lor}\), and the proof is complete. \( \square \)

Now we are ready to state the main result of this section.

**Theorem 4.2.** Let \( I \) be an ideal of \( C_b(X, E)' \) separating the points of \( C_b(X, E) \). Then
\[
(C_b(X, E), |\sigma|(C_b(X, E), I))' = I.
\]

**Proof:** To see that \((C_b(X, E), |\sigma|(C_b(X, E), I))' \subset I \) assume that \( \varphi_{\varphi} \in I \) and \( \rho_{\varphi} \in (C_b(X, E), |\sigma|(C_b(X, E), I))' \). In view of Lemma 2.6 we have to show that \( \in C_b(X, E)' \), that is \( \varphi_{\varphi} \in C_b(X, E)' \) and \( \varphi_{\varphi} \in C_b(X, E)' \). In fact, we know
Lemma 4.4. is locally solid, dense in $(\text{rem } 8.4.1)$ $H$

Proof: Then $m \in M$ is easy to show that $\varphi_m(u) \to 0$. Indeed, $u_0 \otimes e_0 \to 0$ for $|\varphi|((C_b(X), X, E))$, because for each $\varphi \in C_b(X)_I^\prime$, $p_{x_0}^\varphi(u_0 \otimes e_0) = p_{x_0}(u_0)$. Hence by Theorem 4.1 $u_0 \otimes e_0 \to 0$ for $|\varphi|((C_b(X), X, E), I)$. Since $|\varphi_m(u_0)| \leq \varphi_m(|u_0|) = |(u_0 \otimes e_0) = \rho_{x_0}(u_0 \otimes e_0)$, we obtain that $\varphi_m(u_0) \to 0$.

Now let $\in I$. Then for $f \in G((X, E), |(f)| \leq |(f) = \rho_{x_0}(f)$, so is $|\varphi|((C_b(X), X, E), I)$-continuous, i.e., $\in (G((X, E), |\varphi|((C_b(X), X, E), I))^\prime$, as desired.

As an application of Lemma 4.2 we have:

**Corollary 4.3.** Let $I$ be an ideal of $C_b(X, E)^\prime$ separating the points of $C_b(X, E)$. Then for a subset $H$ of $C_b(X, E)$ the following statements are equivalent:

(i) $H$ is bounded for $\sigma(C_b(X, E), I)$;

(ii) $S(H)$ is bounded for $\sigma(C_b(X, E), I)$.

**Proof:** (i) $\implies$ (ii) By Theorem 4.2 and the Mackey theorem (see [Wi, Theorem 8.4.1]) $H$ is bounded for $|\varphi|((C_b(X, E), I)$. Since the topology $|\varphi|((C_b(X, E), I)$ is locally solid, $S(H)$ is bounded for $|\varphi|((C_b(X, E), I)$, hence $S(H)$ is bounded for $\sigma(C_b(X, E), I)$.

(ii) $\implies$ (i) It is obvious.

**Lemma 4.4.** Let $I_\phi = \{ \phi : m \in M_\phi(X, E') \}$, where $\phi = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma(X, E))$ (resp. $\phi = \tau$; resp. $\phi = t$). Then

$$C_b(X)_I^\prime = \{ \varphi_\mu : m \in M_\phi(X) \}.$$ 

**Proof:** Assume that $\varphi \in C_b(X)_I^\prime$, i.e., $\varphi \in C_b(X)^\prime$ and $|\varphi| = \varphi_m$ for some $m \in M_\phi(X, E')$. Then $\varphi = \varphi_\mu$ for some $\mu \in M(X)$, and $|\varphi_\mu| = \varphi_\mu |\mu| \leq \varphi_m = \varphi |\mu|$ (see Lemma 3.4). It follows that $|\mu| \leq |m|$, where $|m| \in M_\phi(X)^\dagger$. Since $M_\phi(X)$ is an ideal of $M(X)$, we get $\mu \in M_\phi(X)$.

Conversely, assume that $\mu \in M_\phi(X)$ and $e_0 \in S_E$ and let $e^* \in E'$ be such that $e^*(e_0) = 1$ and $\|e^*\|_{E'} = 1$. Let us set $m(B) = \mu(B)e^*$ for all $B \in B(X)$. Then $m : B(X) \to E'$ is finitely additive, and for each $e \in E$ we have $u_0(B) = m(B)(e) = (e^*(e_0)\mu)(B)$ for all $B \in B(X)$. Hence $m_{e_0} \in M_\phi(X)$ for each $e \in E$. It is easy to show that $|m| = |\mu| = |\mu| = \varphi_\mu$, so $\varphi_\mu \in C_b(X)_I^\prime$, as desired.

As an application of Lemma 4.1 and Lemma 4.4 we get:
Corollary 4.5. For \( z = \sigma \) and \( C_b(X) \otimes E \) dense in \((C_b(E), \beta_\sigma(X, E))\) (resp. \( z = \tau \); resp. \( z = t \)) we have:
\[
|\sigma|(C_b(X, E), M_z(X, E')) = |\sigma|(C_b(X), M_z(X))
\]
and
\[
|\sigma|(C_b(X, E), M_z(X, E')) = |\sigma|(C_b(X), M_z(X)).
\]

We now define the absolute Mackey topology \(|\tau|((\mathcal{G}(X, E), I))\) on \( C_b(X, E) \) as the topology on uniform convergence on the family of all solid absolutely convex \( \sigma(I, C_b(X, E)) \)-compact subsets of \( I \). In view of Theorem 2.3 \(|\tau|((C_b(X, E), I))\) is a locally convex-solid topology.

The following theorem strengthens the classical Mackey-Arens theorem for the class of locally convex-solid topologies on \( C_b(X, E) \).

Theorem 4.6. Let \( \tau \) be a locally convex-solid topology on \( C_b(X, E) \) and let \((C_b(X, E), \tau)\) be a locally convex-solid topology on \( C_b(X, E) \).

\[
|\sigma|(C_b(X, E), I_{\tau}) \subset \tau \subset |\tau|(C_b(X, E), I_{\tau}).
\]

Proof: To show that \(|\sigma|(C_b(X, E), I_{\tau}) \subset \tau\), assume that \((f_\alpha)\) is a sequence in \( C_b(X, E) \) and \( f_\alpha \xrightarrow{\tau} 0 \). Let \( \epsilon > 0 \) be given. Then there exists a net \((h_\alpha)\) in \( C_b(X, E) \) such that \( \|h_\alpha\| \leq \|f_\alpha\| \) and \( \rho_\Phi(f_\alpha) = |\|f_\alpha\| - |(h_\alpha)|| \leq \epsilon \). Since \( \tau \) is locally solid, \( h_\alpha \xrightarrow{\tau} 0 \). Hence \( h_\alpha \xrightarrow{\tau} 0 \), because \( \sigma(C_b(X, E), I_{\tau}) \subset \tau \). Thus \( \rho_\Phi(f_\alpha) \xrightarrow{\tau} 0 \), and this means that \( f_\alpha \xrightarrow{\tau} 0 \) for \(|\sigma|(C_b(X, E), I_{\tau})\).

Now we show that \( \tau \subset |\tau|(C_b(X, E), I_{\tau}) \). Indeed, let \( B_\tau \) be a local base at zero for \( \tau \) consisting of solid absolutely convex and \( \tau \)-closed sets and let \( V \in B_\tau \). Then by Theorem 2.3 and the Banach-Alaoglu’s theorem, \( V^0 \) is a solid absolutely convex and \( \sigma(I_{\tau}, C_b(X, E)) \)-compact subset of \( I_{\tau} \). Hence
\[
0(V^0) = \overline{\text{abs conv} V^\sigma} = \overline{\text{abs conv} V^\tau} = V,
\]
so \( \tau \) is the topology of uniform convergence on the family \( \{V^0 : V \in B_\tau\} \). It follows that \( \tau \subset |\tau|(C_b(X, E), I_{\tau}) \).

Corollary 4.7. Let \( I_z = \{ m : m \in M_z(X, E') \} \), where \( z = \sigma \) and \( C_b(X) \otimes E \) is dense in \((C_b(X, E), \beta_\sigma(X, E))\) (resp. \( z = \tau \) and \( X \) is paracompact; resp. \( z = t \) and \( X \) is a \( P \)-space). Then
\[
\beta_z(X, E) = |\tau|(C_b(X, E), M_z(X, E')) = \tau(C_b(X, E), M_z(X, E')),
\]
and for a locally convex-solid topology \( \tau \) on \( C_b(X, E) \) with \( C_b(X, E)_{\tau} = I_z \) we have:
\[
|\sigma|(C_b(X, E), M_z(X, E')) \subset \tau \subset \beta_z(X, E).
\]
Duality theory of spaces of vector-valued continuous functions

Proof: It is known that under our assumptions $\beta_z(X,E)$ is a Mackey topology (see [KhO1, Corollary 6] for $z = \sigma$, [Kh, Theorem 6.2] for $z = \tau$ and [Kh, Theorem 5] for $z = t$). Hence $\tau(C_b(X,E), M_z(X,E')) = \beta_z(X,E)$. On the other hand, since $\beta_z(X,E)$ is a locally convex-solid topology and $(C_b(X,E), \beta_z(X,E))' = I_z$, by Corollary 4.6 we get $\beta_z(X,E) \subseteq |\tau|(C_b(X,E), M_z(X,E'))$. □

References


Faculty of Mathematics, Informatics and Econometrics, University of Zielona Góra, ul. Szafrana 4a, 65–516 Zielona Góra, Poland

E-mail: M.Nowak@wmie.uz.zgora.pl

(Received October 14, 2003, revised September 27, 2004)