Complete hypersurfaces with constant scalar curvature in a sphere

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Abstract. In this paper, by using Cheng-Yau’s self-adjoint operator $\Box$, we study the complete hypersurfaces in a sphere with constant scalar curvature.

Keywords: hypersurface, sphere, scalar curvature

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1. Introduction

Let $S^{n+1}$ be an $(n+1)$-dimensional unit sphere with constant sectional curvature 1, let $M^n$ be an $n$-dimensional hypersurface in $S^{n+1}$, and $e_1, \ldots, e_n$ a local orthonormal frame field on $M^n$, $\omega_1, \ldots, \omega_n$ its dual coframe field. Then the second fundamental form of $M^n$ is

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j.$$  \hspace{1cm} (1)

Further, near any given point $p \in M^n$, we can choose a local frame field $e_1, \ldots, e_n$ so that at $p$, $\sum_{i,j} h_{ij} \omega_i \otimes \omega_j = \sum_i k_i \omega_i \otimes \omega_i$. Then the Gauss equation says

$$R_{ijij} = 1 + k_i k_j, \hspace{0.5cm} i \neq j.$$ \hspace{1cm} (2)

$$n(n-1)(R-1) = n^2 H^2 - |h|^2,$$ \hspace{1cm} (3)

where $R$ is the normalized scalar curvature, $H = \frac{1}{n} \sum_i k_i$ the mean curvature and $|h|^2 = \sum_i k_i^2$ the norm square of the second fundamental form of $M^n$.

As it is well known, there are many rigidity results for minimal hypersurfaces or hypersurfaces with constant mean curvature $H$ in $S^{n+1}$ by use of J. Simons’ method, for example, see [1], [3], [4], [6], [9], etc.

On the other hand, Cheng-Yau [2] introduced a new self-adjoint differential operator $\Box$ to study the hypersurfaces with constant scalar curvature. Later, Li [5] obtained interesting rigidity results for hypersurfaces with constant scalar curvature in space-forms using the Cheng-Yau’s self-adjoint operator $\Box$.

In the present paper, we use Cheng-Yau’s self-adjoint operator $\Box$ to study the complete hypersurfaces in a sphere with constant scalar curvature, and prove the following theorem:
Theorem. Let $M^n$ be an $n$-dimensional ($n \geq 3$) complete hypersurface with constant normalized scalar curvature $R$ in $S^{n+1}$. If

1. $\bar{R} = R - 1 \geq 0$,
2. the mean curvature $H$ of $M^n$ satisfies

$$\bar{R} \leq \sup H^2 \leq \frac{1}{n^2} \left( (n-1)^2 \frac{n\bar{R} + 2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R} + 2} \right),$$

then either

$$\sup H^2 = \bar{R}$$

and $M^n$ is a totally umbilical hypersurface; or

$$\sup H^2 = \frac{1}{n^2} \left( (n-1)^2 \frac{n\bar{R} + 2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R} + 2} \right),$$

and $M^n = S^1(\sqrt{1-t^2}) \times S^{n-1}(r)$, $r = \sqrt{\frac{n-2}{n\bar{R}+1}}$.

2. Preliminaries

Let $M^n$ be an $n$-dimensional complete hypersurface in $S^{n+1}$. We choose a local orthonormal frame $e_1, \ldots, e_{n+1}$ in $S^{n+1}$ such that at each point of $M^n$, $e_1, \ldots, e_n$ span the tangent space of $M^n$ and form an orthonormal frame there. Let $\omega_1, \ldots, \omega_{n+1}$ be its dual coframe. In this paper, we use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+1; \quad 1 \leq i, j, k, \ldots \leq n.$$

Then the structure equations of $S^{n+1}$ are given by

1. $d\omega_A = \sum_B \omega_{AB} \wedge \omega_B$, $\omega_{AB} + \omega_{BA} = 0$,
2. $d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D$,
3. $K_{ABCD} = (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC})$.

Restricting these forms to $M^n$, we have

4. $\omega_{n+1} = 0$.

From Cartan’s lemma we can write

5. $\omega_{n+1} = \sum_j h_{ij} \omega_j$, $h_{ij} = h_{ji}$. 
From these formulas, we obtain the structure equations of $M^n$:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

where $R_{ijkl}$ are the components of the curvature tensor of $M^n$ and

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$$

is the second fundamental form of $M^n$. We also have

$$R_{ij} = (n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj},$$

$$n(n-1)(R-1) = n^2H^2 - |h|^2,$$

where $R$ is the normalized scalar curvature, and $H$ the mean curvature.

Define the first and the second covariant derivatives of $h_{ij}$, say $h_{ijk}$ and $h_{ijkl}$ by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj},$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mj}\omega_{mi} + \sum_m h_{im}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}.$$

Then we have the Codazzi equation

$$h_{ijk} = h_{ikj},$$

and the Ricci’s identity

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}.$$

For a $C^2$-function $f$ defined on $M^n$, we define its gradient and Hessian ($f_{ij}$) by the following formulas

$$df = \sum_i f_i \omega_i, \quad \sum_j f_{ij} \omega_j = df + \sum_j f_j \omega_{ji}. $$
The Laplacian of \( f \) is defined by \( \Delta f = \sum_i f_{ii} \).

Let \( \phi = \sum_{ij} \phi_{ij} \omega_i \otimes \omega_j \) be a symmetric tensor defined on \( M^n \), where
\[
(20) \quad \phi_{ij} = nH \delta_{ij} - h_{ij}.
\]

Following Cheng-Yau [2], we introduce the operator \( \Box \) associated to \( \phi \) acting on any \( C^2 \)-function \( f \) by
\[
(21) \quad \Box f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH \delta_{ij} - h_{ij}) f_{ij}.
\]

Since \( \phi_{ij} \) is divergence-free, it follows [2] that the operator \( \Box \) is self-adjoint relative to the \( L^2 \) inner product of \( M^n \), i.e.
\[
(22) \quad \int_{M^n} f \Box g = \int_{M^n} g \Box f.
\]

We can choose a local frame field \( e_1, \ldots e_n \) at any point \( p \in M^n \), such that \( h_{ij} = k_i \delta_{ij} \) at \( p \), and by use of (21) and (14), we have
\[
(23) \quad \Box (nH) = nH \Delta (nH) - \sum_i k_i (nH)_{ii}
\]
\[
= \frac{1}{2} \Delta (nH)^2 - \sum_i (nH)_{ii}^2 - \sum_i k_i (nH)_{ii}
\]
\[
= \frac{1}{2} n(n-1) \Delta R + \frac{1}{2} \Delta |h|^2 - n^2 |\nabla H|^2 - \sum_i k_i (nH)_{ii}.
\]

On the other hand, through a standard calculation by use of (17) and (18), we get
\[
(24) \quad \frac{1}{2} \Delta |h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i k_i (nH)_{ii} + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2.
\]

Putting (24) into (23), we have
\[
(25) \quad \Box (nH) = \frac{1}{2} n(n-1) \Delta R + |\nabla h|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2.
\]

From (11), we have \( R_{ijij} = 1 + k_i k_j, \ i \neq j \), and by putting this into (25), we obtain
\[
(26) \quad \Box (nH) = \frac{1}{2} n(n-1) \Delta R + |\nabla h|^2 - n^2 |\nabla H|^2 + n|h|^2 - n^2 H^2 - |h|^4 + nH \sum_i k_i^2.
\]
Let $\mu_i = k_i - H$ and $|Z|^2 = \sum_i \mu_i^2$. We have

$$\sum_i \mu_i = 0, \quad |Z|^2 = |h|^2 - nH^2,$$

$$\sum_i k_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3.$$  \hfill (27)

From (26)–(28), we get

$$\Box(nH) = \frac{1}{2} n(n-1) \Delta R + |\nabla h|^2 - n^2 |\nabla H|^2$$

$$+ |Z|^2 (n + nH^2 - |Z|^2) + nH \sum_i \mu_i^3.$$  \hfill (29)

We need the following algebraic lemma due to M. Okumura [7] (see also [1]).

**Lemma 2.1.** Let $\mu_i$, $i = 1, \ldots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta = \text{constant} \geq 0$. Then

$$- \frac{n - 2}{\sqrt{n(n-1)}} \beta^3 \leq \sum_i \mu_i^3 \leq \frac{n - 2}{\sqrt{n(n-1)}} \beta^3,$$  \hfill (30)

and the equality holds in (30) if and only if at least $(n - 1)$ of the $\mu_i$ are equal.

By use of Lemma 2.1, we have

$$\Box(nH) \geq \frac{1}{2} n(n-1) \Delta R + |\nabla h|^2 - n^2 |\nabla H|^2$$

$$+ (|h|^2 - nH^2)(n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{|h|^2 - nH^2}).$$  \hfill (31)

3. **Proof of Theorem**

The following lemma is essentially due to Cheng-Yau [2] (see also [5]).

**Lemma 3.1.** Let $M$ be an $n$-dimensional hypersurface in $S^{n+1}$. Suppose that the normalized scalar curvature $R = \text{constant}$ and $R \geq 1$. Then $|\nabla h|^2 \geq n^2 |\nabla H|^2$.

From the assumption of Theorem that $R$ is constant and $\bar{R} = R - 1 \geq 0$ and Lemma 3.1 we have

$$\Box(nH) \geq (|h|^2 - nH^2)(n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{|h|^2 - nH^2}).$$  \hfill (32)
By Gauss equation (14) we know that
\begin{equation}
|Z|^2 = |h|^2 - nH^2 = \frac{n-1}{n}(|h|^2 - n\bar{R}).
\end{equation}

From (32) and (33) we have
\begin{equation}
\Box(nH) \geq \frac{n-1}{n}(|h|^2 - n\bar{R})\phi_H(|h|),
\end{equation}
where
\[
\phi_H(|h|) = n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{|h|^2 - nH^2}.
\]

By (33) we can write \(\phi_H(|h|)\) as
\begin{equation}
\phi_H(|h|) = n + 2(n-1)\bar{R} - \frac{n-2}{n}|h|^2 - \frac{n-2}{n}\sqrt{(n(n-1)R + |h|^2)(|h|^2 - n\bar{R})}.
\end{equation}

Therefore (34) becomes
\begin{equation}
\Box(nH) \geq \frac{n-1}{n}(|h|^2 - n\bar{R})\phi_H(|h|).
\end{equation}

It is a direct check that our assumption
\[
\sup H^2 \leq \frac{1}{n^2}\left((n-1)^2\frac{n\bar{R} + 2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R} + 2}\right)
\]
is equivalent to
\begin{equation}
\sup |h|^2 \leq \frac{n}{(n-2)(n\bar{R} - 2)}\left[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\right],
\end{equation}
i.e.
\begin{equation}
(n + 2(n-1)\bar{R} - \frac{n-2}{n}\sup |h|^2)^2
\end{equation}
\[
\geq \frac{(n-2)^2}{n^2}(n(n-1)\bar{R} + \sup |h|^2)(\sup |h|^2 - n\bar{R}).
\]

But it is clear from (37) that (38) is equivalent to
\begin{equation}
(n + 2(n-1)\bar{R} - \frac{n-2}{n}\sup |h|^2)
\end{equation}
\[
\geq \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + \sup |h|^2)(\sup |h|^2 - n\bar{R})}.
\]
So under the hypothesis that
\[
\sup H^2 \leq \frac{1}{n^2} \left[ (n-1)^2 \frac{n\bar{R} + 2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R} + 2} \right],
\]
we have
\[
(40) \quad \phi_R(\sqrt{\sup |h|^2}) \geq 0.
\]

On the other hand,
\[
\Box(nH) = \sum_{i,j} (nH\delta_{ij} - nh_{ij})(nH)_{ij} = \sum_{i} (nH - nh_{ii})(nH)_{ii}
\]
\[
= n \sum_{i} H(nH)_{ii} - n \sum_{i} k_i(nH)_{ii} \leq (|H|_{\text{max}} - C) \Delta(nH),
\]
where $|H|_{\text{max}}$ is the maximum of the mean curvature $H$ and $C = \min k_i$ is the minimum of the principal curvatures of $M^n$.

Now we need the following maximum principle at infinity for complete manifolds due to Omori [8] and Yau [10]:

**Lemma 3.2.** Let $M^n$ be an $n$-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $f : M^n \to \mathbb{R}$ a smooth function bounded from below. Then for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M^n$ such that

(i) $|\nabla f|(p_\varepsilon) < \varepsilon$,

(ii) $\Delta f(p_\varepsilon) > -\varepsilon$,

(iii) $\inf f \leq f(p_\varepsilon) \leq \inf f + \varepsilon$.

Since the scalar curvature of $M$ is a constant, from the hypothesis that $\bar{R} \leq \sup H^2 \leq \frac{1}{n^2} \left[ (n-1)^2 \frac{n\bar{R} + 2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R} + 2} \right]$, and Gauss equation (14), we know the squared norm $|h|^2$ of the second fundamental form is bounded from above, from (11) we know that the sectional curvature is bounded from below. So we may apply Lemma 3.2 to the smooth function $f$ on $M^n$ defined by

\[
f = \frac{1}{\sqrt{1 + (nH)^2}}.
\]

It is immediate to check that
\[
|\nabla f|^2 = \frac{1}{4} \frac{|\nabla (nH)^2|^2}{(1 + (nH)^2)^3}
\]
and that
\[
\Delta f = -\frac{1}{2} \frac{\Delta (nH)^2}{(1 + (nH)^2)^{3/2}} + \frac{3}{4} \frac{|\nabla (nH)^2|^2}{(1 + (nH)^2)^{5/2}}.
\]
By Lemma 3.2 we can find a sequence of points \( p_k \), \( k \in N \) in \( M^n \), such that

\[
\lim_{k \to \infty} f(p_k) = \inf f, \quad \Delta f(p_k) > -\frac{1}{k}, \quad |\nabla f|^2(p_k) < \frac{1}{k^2}.
\]

Using (44) in equations (42) and (43) and the fact that

\[
\lim_{k \to \infty} (nH)(p_k) = \sup_{p \in M^n} (nH)(p),
\]

we get

\[
-\frac{1}{k} \leq -\frac{1}{2}(1 + (nH)^2)^{3/2} - \frac{3}{k^2}(1 + (nH)^2)^{1/2}.
\]

Hence we obtain

\[
\frac{\Delta(nH)^2}{(1 + (nH)^2)^2}(p_k) < \frac{2}{k} \left( \frac{1}{\sqrt{1 + (nH)^2(p_k)}} + \frac{3}{k} \right).
\]

On the other hand, by (36) and (41), we have

\[
\frac{n-1}{n}(|h|^2 - n\bar{R})\phi_R(|h|) \leq \square(nH) \leq n(|H|_{\max} - C)\Delta(nH).
\]

At points \( p_k \) of the sequence given in (44), this becomes

\[
\frac{n-1}{n}(|h|^2(p_k) - n\bar{R})\phi_R(|h|(p_k)) \leq \square(nH(p_k)) \leq n(|H|_{\max} - C)\Delta(nH(p_k)).
\]

Letting \( k \to \infty \) and using (47) we have that the right hand side of (49) goes to zero, so we have either \( \frac{n-1}{n}(\sup |h|^2 - n\bar{R}) = 0 \), i.e. \( \sup H^2 = \bar{R} \), or \( \phi_R(\sqrt{\sup |h|^2}) = 0 \).

If \( \sup |h|^2 = n\bar{R} \), by (33) \( |Z|^2 = \frac{n-1}{n}(|h|^2 - n\bar{R}) \) we have

\[
\sup |Z|^2 = \frac{n-1}{n}(\sup |h|^2 - n\bar{R}) = 0, \text{ hence } |Z|^2 = 0 \text{ and } M^n \text{ is totally umbilical.}
\]

If \( \phi_R(\sqrt{\sup |h|^2}) = 0 \), it is easy to prove that

\[
\sup H^2 = \frac{1}{n^2}[(n-1)^2 + \frac{2(n-1)}{n-2} - 2(n-1) + \frac{n-2}{n-2}], \text{ hence equalities hold in (30) and Lemma 3.1, and it follows that } k_i = \text{constant for all } i \text{ and } (n-1) \text{ of the } k_i \text{'s are equal. After remuneration if necessary, we can assume that}
\]

\[
k_1 = k_2 = \cdots = k_{n-1}, \quad k_1 \neq k_n.
\]

Therefore, \( M^n \) is a isoparametric hypersurface in \( S^{n+1} \) with two distinct principal curvatures, hence \( M^n = S^1(\sqrt{1 - r^2}) \times S^{n-1}(r), k_1 = \cdots = k_{n-1} = \sqrt{1 - r^2}/r, k_1 = -r/\sqrt{1 - r^2}. \) From (14), it is easy to see that \( n(n-1)\bar{R} = (n-1)(n-2 - n^2)/r^2, \) thus \( r = \sqrt{\frac{n-2}{n(n+1)}}. \) This completes the proof of Theorem.

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