On subspaces of pseudo-radial spaces

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Abstract. It is proved that, under the Martin's Axiom, every T_1-space with countable tightness is a subspace of some pseudo-radial space. We also give several characterizations of subspaces of pseudo-radial spaces and conclude that being a subspace of a pseudo-radial space is a local property.

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1. Introduction.

In [1] the authors proposed the following problem: find necessary or sufficient (or both) conditions for a topological space to be a subspace of a pseudo-radial space. They also asked whether, in particular, $N \cup \{p\}$ is a subspace of a pseudo-radial space for $p \in \beta N \setminus N$. In Section 2 we give some necessary and sufficient conditions for a space to be a subspace of a pseudo-radial space. In Section 3 we prove that, under Martin's Axiom, every T_1 space with countable tightness is a subspace of a pseudo-radial space. Thus we partly answer the question 3.4 of [1].

Definition 1.1. *A subset A of a topological space X is called closed w.r.t. chain-net if for each $x \in X$, if there exists a transfinite sequence in A converging to x, then $x \in A$. For any $B \subseteq X$ we denote by $clseq_X B$ the smallest subset of X containing B and closed w.r.t. chain-net.*

Definition 1.2 (5)**.** *A space is called pseudo-radial if for each $A \subseteq X$, $\overline{A} = clseq_X A$. A space is called sub pseudo-radial if it is a subspace of some pseudo-radial space.*

There was a lot of equivalent definitions of pseudo-radial spaces (see [1] and [2]).

All spaces are assumed to be T_1. If $\{X_\alpha : \alpha \in \Sigma\}$ is a family of spaces, we denote by $\oplus_{\alpha\in\Sigma} X_\alpha$ the topological sum of $\{X_\alpha : \alpha \in \Sigma\}$.

2. Some characterizations.

We start with a lemma.

Lemma 2.1. *Every quotient of a sub pseudo-radial space is sub pseudo-radial.*

Proof: Since every quotient of a pseudo-radial space is pseudo-radial, it is enough to see that for any class M of spaces, if M is closed under quotient mappings, then the class consisting of subspaces of the spaces in M is also closed under quotient mappings. □

We call a space a prime space if it has only one non-isolated point. Given any space X and a point p in X, denote by X_p the prime space constructed by making each point, other than p, isolated with p retaining its original neighborhoods. We call X_p the prime factor of X at p. Obviously, each topological space is the quotient of the topological sum of all its prime factors.

Proposition 2.2. *For a space X the following conditions are equivalent:*

(i) *X is sub pseudo-radial,*
(ii) *for every p in X, X_p is sub pseudo-radial,*
(iii) *for each subset A of X and $q \in \overline{A}$, there exists a subset B of A such that $q \in \overline{B}$ and $B \cup \{q\}$ is sub pseudo-radial.*

PROOF: The implication (i) → (iii) is obvious. The proof of the implication (i) → (ii) is completely the same as that of Proposition 5.1 of [3].

To prove the left two implications, let $Z = \oplus_{p \in X} X_p$ when (ii) holds and $Z = \oplus\{Y : Y \subseteq X$ and Y is sub pseudo-radial$\}$ when (iii) holds. It is easy to see that, in both cases, X is a quotient of Z and Z is sub pseudo-radial. By virtue of Lemma 2.1, X is a pseudo-radial space when (ii) or (iii) holds. □

Corollary 2.3. *A space X is sub pseudo-radial if either*

(i) *each subset of X with cardinality not greater than the tightness of X is sub pseudo-radial, or*
(ii) *each point of X has a sub pseudo-radial neighborhood.*

3. Countable case.

In this section, N denotes the set of natural numbers. βN is the Čech-Stone compactification of the discrete space N. If A and B are subsets of N, $A \subseteq {}^*B$ means that there exists an n in N such that $A \setminus \{0, 1, 2, \dots, n-1\} \subseteq B$. A family $\mathcal{A}$ of subsets of N is called an almost disjoint family, shortened as a.d. family, if for any distinct elements A_1 and A_2 of $\mathcal{A}$, $A_1 \cap A_2$ is finite. We say that $\mathcal{A}$ has sfip (strong finite intersection property) if every nonempty finite subfamily of $\mathcal{A}$ has infinite intersection. We say that B is a pseudo-intersection of $\mathcal{A}$ if $B \subseteq {}^*A$ for each A in $\mathcal{A}$. For any set A, $|A|$ denotes the cardinality of A; c denotes the cardinality of the power set $\mathcal{P}N$ of N.

The following lemma is well-known in set-theory (for example, see 11C of [14]).

Lemma 3.1 (MA). *For each family $\mathcal{A}$ of subsets of N, if $|\mathcal{A}| < c$ and $\mathcal{A}$ has sfip, then $\mathcal{A}$ has an infinite pseudo-intersection.*

Theorem 3.2 (MA). *Every space with countable tightness is sub pseudo-radial.*

PROOF: It is a consequence of the following Theorem 3.3 and (i) of Corollary 2.3, □

Theorem 3.3 (MA). *Every countable space is sub pseudo-radial.*

PROOF: By virtue of Proposition 2.2, we only need to prove that every countable prime space is sub pseudo-radial. Let $X = N \cup \{p\}$ be a prime space with the unique non-isolated point p. We prove the X is sub pseudo-radial.

W.l.o.g., we assume that $\chi(p, X) = c$. Let $\mathcal{B}$ be a filter base on N such that the set $\{B \cup \{p\} : B \in \mathcal{B}\}$ constitutes a local base at p. Let $\mathcal{A} = \mathcal{B} \cup \{A \subseteq N;\ p \in \overline{A}^X$ and A contains no infinite pseudo-intersection of $\mathcal{B}\}$.

Let $\mathcal{A} = \{A_\alpha : \alpha < c\}$ be an enumeration of $\mathcal{A}$ such that for each $A \in \mathcal{A}$, the set $\{\alpha < c : A_\alpha = A\}$ is unbounded in c. We construct by induction an almost disjoint sequence $\mathcal{C} = \{C_\alpha : \alpha < c\}$ and a sequence $\{B_\alpha : \alpha < c\} \subseteq \mathcal{B}$ such that

(i) $\forall\, \alpha < c$, $C_\alpha \subseteq A_\alpha$ and C_α is infinite;
(ii) $\forall\, \beta < \alpha < c$, if $A_\beta \in \mathcal{B}$, then $C_\alpha \subseteq^* A_\beta$;
(iii) $\forall\, \alpha < c$, $C_\alpha \cap B_\alpha = \emptyset$.

Assume $\alpha < c$ and we have constructed $\{C_\beta : \beta < \alpha\}$ and $\{B_\beta : \beta < \alpha\}$ satisfying (i) to (iii). We construct C_α, B_α as follows.

Case I. $A_\alpha \notin \mathcal{B}$. Since $p \in \overline{A}_\alpha^X$, we apply Lemma 3.1 on the family

$$\mathcal{B}' = \{B_\beta \cap A_\alpha : \beta < \alpha\} \cup \{A_\beta \cap A_\alpha : \beta \le \alpha \text{ and } A_\beta \in \mathcal{B}\}.$$

We obtain an infinite subset A of A_α which is a pseudo-intersection of $\mathcal{B}'$. Since A cannot be a pseudo-intersection of $\mathcal{B}$, there is a $B \in \mathcal{B}$ such that $A \setminus B$ is infinite. Let $C_\alpha = A \setminus B$ and $B_\alpha = B$.

Case II. $A_\alpha \in \mathcal{B}$. Let $\mathcal{B}'$ as in the Case I. Since X is a T_1 space and $|\mathcal{B}'| < c = \chi(p, X)$, there exists a $B^* \in \mathcal{B}$ such that for each finite subfamily $\mathcal{B}'$ of $\mathcal{B}$, $\bigcap_{B \in \mathcal{B}'} B \setminus B^*$ is infinite. Therefore the family $\mathcal{F} = \{B \setminus B^* : B \in \mathcal{B}\}$ has the sfip. Again by Lemma 3.1, we obtain an infinite $A \subseteq A_\alpha \setminus B^*$ which is a pseudo-intersection of $\mathcal{B}'$. Let $C_\alpha = A$ and $B_\alpha = B^*$. Thus we have finished the induction.

Now we construct a Hausdorff pseudo-radial space Y containing X as a subspace. Let $Y = X \cup (c \times \{0\})$. We define a topology on Y as follows. The set N is open discrete in Y. For each $\alpha < c$, let $\{C_\alpha \setminus n \cup \{(\alpha, 0)\} : n \in N\}$ be a local base at the point $(\alpha, 0)$. For the point p, let $\{U(A_\alpha) : A_\alpha \in \mathcal{B},\ \alpha < c\}$ be a local base, where $U(A_\alpha) = \{p\} \cup A_\alpha \cup \{(\beta, 0) : \alpha < \beta < c\}$. It is easy to see that the above topology is well-defined and that X is a subspace of Y. Y is Hausdorff because of the above property (iii) and the fact that, for each B_α, the set $\{\beta < c : A_\beta = B_\alpha\}$ is unbounded in c. We are left to check that Y is pseudo-radial. Let $E \subseteq Y$ and $y \in \overline{E}^Y$. To avoid the trivialities, we assume $y = p$ and $E \subseteq N$. Then $p \in \overline{E}^X$. If $E \in \mathcal{A}$, then $\{(\alpha, 0) : \alpha < c \text{ and } A_\alpha = E\} \subseteq clseq_Y E$. Since the set $\{\alpha < c : A_\alpha = E\}$ is unbounded in c, $p \in clseq_Y\{(\alpha, 0) : A_\alpha = E\}$. Thus $p \in clseq_Y E$. If $E \notin \mathcal{A}$, then there exists an infinite subset E' of E which is a pseudo-intersection of $\mathcal{B}$. But this obviously implies that $p \in clseq_X E$. Therefore $p \in clseq_Y E$. We are done. $\square$

Remark. For any $p \in \beta N \setminus N$, it is easy to see that $N \cup \{p\}$ is not pseudo-radial. But by Theorem 3.2, we see that it is sub pseudo-radial under the Martin's Axiom. Thus we partly answer the question 4 of [1].

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