Research Article

Starlike and Convex Properties for Hypergeometric Functions

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The purpose of the present paper is to give some characterizations for a (Gaussian) hypergeometric function to be in various subclasses of starlike and convex functions. We also consider an integral operator related to the hypergeometric function.

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1. Introduction

Let \( \mathcal{T} \) be the class consisting of functions of the form

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0,
\]

that are analytic and univalent in the open unit disk \( U = \{ z : |z| < 1 \} \). Let \( \mathcal{T}^*(\alpha) \) and \( \mathcal{C}(\alpha) \) denote the subclasses of \( \mathcal{T} \) consisting of starlike and convex functions of order \( \alpha \) \((0 \leq \alpha < 1)\), respectively [1].

Recently, Bharati et al. [2] introduced the following subclasses of starlike and convex functions.

**Definition 1.1.** A function \( f \) of the form (1.1) is in \( S_p T(\alpha, \beta) \) if it satisfies the condition

\[
\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} \geq a \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \quad a \geq 0, \quad 0 \leq \beta < 1,
\]

and \( f \in \mathcal{UC}(\alpha, \beta) \) if and only if \( zf' \in S_p \mathcal{T}(a, \beta) \).
Definition 1.2. A function $f$ of the form (1.1) is in $P\mathcal{T}(a)$ if it satisfies the condition

$$\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} + \alpha \geq \left| \frac{zf'(z)}{f(z)} - a \right|, \quad 0 < a < \infty,$$

(1.3)

and $f \in C\mathcal{P}\mathcal{T}(a)$ if and only if $zf' \in \mathcal{P}\mathcal{T}(a)$.

Bharati et al. [2] showed that

$S_p\mathcal{T}(a, \beta) = \mathcal{T}^*((a+\beta)/(1+a))$, $\mathcal{UCT}(a, \beta) = C((a+\beta)/(1+a))$, $\mathcal{P\mathcal{T}}(a) = \mathcal{T}^*(1-a)$ $(0 < a \leq 1)$, and $C\mathcal{P}\mathcal{T}(a) = C(1-a)$ $(0 < a \leq 1)$. In particular, we note that $\mathcal{UCT}(1,0)$ is the class of uniformly convex functions given by Goodman [3] (also see [4–6]).

Let $F(a, b; c; z)$ be the (Gaussian) hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \left( \begin{array}{c} a \beta \cdots \beta \\ c \end{array} \right)_n \frac{a_n}{(1)_n} z^n,$$

(1.4)

where $c \neq 0, -1, -2, \ldots$, and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1 + \lambda) \cdots (\lambda + n - 1) & \text{if } n \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

(1.5)

We note that $F(a, b; c; 1)$ converges for $\text{Re}(c - a - b) > 0$ and is related to the Gamma function by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$  

(1.6)

Silverman [7] gave necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $\mathcal{T}^*(a)$ and $C(a)$, and also examined a linear operator acting on hypergeometric functions. For the other interesting developments for $zF(a, b; c; z)$ in connection with various subclasses of univalent functions, the readers can refer to the works of Carlson and Shaffer [8], Merkes and Scott [9], and Ruscheweyh and Singh [10].

In the present paper, we determine necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $S_p\mathcal{T}(a, \beta)$, $\mathcal{UCT}(a, \beta)$, $\mathcal{P}\mathcal{T}(a)$, and $C\mathcal{P}\mathcal{T}(a)$. Furthermore, we consider an integral operator related to the hypergeometric function.

2. Results

To establish our main results, we need the following lemmas due to Bharati et al. [2].

Lemma 2.1. (i) A function $f$ of the form (1.1) is in $S_p\mathcal{T}(a, \beta)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} (n(1+a)-(a+\beta))a_n \leq 1-\beta.$$  

(2.1)

(ii) A function $f$ of the form (1.1) is in $\mathcal{UCT}(a, \beta)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} n(n+1-(a+\beta))a_n \leq 1-\beta.$$  

(2.2)
Lemma 2.2. (i) A function \( f \) of the form (1.1) is in \( \mathcal{PT}(a) \) if and only if it satisfies

\[
\sum_{n=2}^{\infty} (n-1+a)a_n \leq a.
\] (2.3)

(ii) A function \( f \) of the form (1.1) is in \( \mathcal{CPPT}(a) \) if and only if it satisfies

\[
\sum_{n=2}^{\infty} n(n-1+a)a_n \leq a.
\] (2.4)

Theorem 2.3. (i) If \( a, b > -1, \ c > 0, \) and \( ab < 0, \) then \( zF(a, b; c; z) \) is in \( \mathcal{SPPT}(\alpha, \beta) \) if and only if

\[
c \geq a + b + 1 - \frac{(1+a)ab}{1-\beta}.
\] (2.5)

(ii) If \( a, b > 0 \) and \( c > a + b + 1, \) then \( F_1(a, b; c; z) = z(2 - F(a, b; c; z)) \) is in \( \mathcal{SPPT}(\alpha, \beta) \) if and only if

\[
\frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \left( 1 + \frac{(1+a)ab}{(1-\beta)(c-a-b-1)} \right) \leq 2.
\] (2.6)

Proof. (i) Since

\[
zF(a, b; c; z) = z + \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n = z - \left| \frac{ab}{c} \right| \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n,
\] (2.7)

according to (i) of Lemma 2.1, we must show that

\[
\sum_{n=2}^{\infty} \left( n(1+a) - (\alpha + \beta) \right) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left| \frac{c}{ab} \right| (1-\beta).
\] (2.8)

Noting that \( (\lambda)_n = \lambda(\lambda + 1)_{n-1} \) and then applying (1.6), we have

\[
\sum_{n=0}^{\infty} \left( (n+2)(1+a) - (\alpha + \beta) \right) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} = (1+a) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1-\beta) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}
\]

\[
= (1+a) \frac{\Gamma(c+1) \Gamma(c-a-b-1)}{\Gamma(c-a) \Gamma(c-b)} + (1-\beta) \frac{c}{ab} \left( \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} - 1 \right).
\] (2.9)
Hence, (2.8) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left(1 + \alpha + (1 - \beta) \frac{c - a - b - 1}{ab}\right) \leq (1 - \beta) \left(\frac{c}{[ab]} + \frac{c}{ab}\right) = 0. \quad (2.10)$$

Thus, (2.10) is valid if and only if $1 + \alpha + (1 - \beta)(c - a - b - 1)/(ab) \leq 0$ or, equivalently, $c \geq a + b + 1 - (1 + a)ab/(1 - \beta)$.

(ii) Since

$$F_1(a, b; c, z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n, \quad (2.11)$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} (n(1 + \alpha) - (\alpha + \beta)) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq 1 - \beta. \quad (2.12)$$

Now,

$$\sum_{n=2}^{\infty} (n(1 + \alpha) - (\alpha + \beta)) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = (1 + \alpha) \sum_{n=1}^{\infty} \frac{n(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}} (1 - \beta) \sum_{n=1}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}$$

$$= \frac{(1 + \alpha)ab}{c} \sum_{n=1}^{\infty} \frac{(a + 1)_{n-1}(b + 1)_{n-1}}{(c + 1)_{n-1}(1)_{n-1}} + (1 - \beta) \sum_{n=1}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{(1 + \alpha)ab}{c} + 1 - \beta\right) - (1 - \beta). \quad (2.13)$$

But this last expression is bounded above by $1 - \beta$ if and only if (2.6) holds.

\[ \square \]

**Theorem 2.4.** (i) If $a, b > -1$, $ab < 0$, and $c > a + b + 2$, then $zF(a, b; c; z)$ is in $\mathcal{U}\mathcal{C}(\alpha, \beta)$ if and only if

$$(1 + \alpha)(a)_{2}(b)_{2} + (3 + 2\alpha - \beta)ab(c - a - b - 2) + (1 - \beta)(c - a - b - 2)_{2} \geq 0. \quad (2.14)$$

(ii) If $a, b > 0$ and $c > a + b + 2$, then $F_1(a, b; c, z) = z(2 - F(a, b; c; z))$ is in $\mathcal{U}\mathcal{C}(\alpha, \beta)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{(1 + \alpha)(a)_{2}(b)_{2}}{(1 - \beta)(c - a - b - 2)_{2}} + \left(\frac{3 + 2\alpha - \beta}{1 - \beta}\right)\left(\frac{ab}{c - a - b - 1}\right) + 1\right) \leq 2. \quad (2.15)$$
Proof. (i) Since $zF$ has the form (2.7), we see from (ii) of Lemma 2.1 that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} n(n+\alpha) - (\alpha + \beta) \frac{(a+1)n(b+1)n}{(c+1)n(1)n+1} \leq \frac{c}{|ab|}(1-\beta). \quad (2.16)$$

Writing $(n+2)((n+2)(1+\alpha) - (\alpha + \beta)) = (1+\alpha)(n+1)^2 + (2+\alpha - \beta)(n+1) + (1-\beta)$, we see that

$$\sum_{n=0}^{\infty} (n+2)((n+2)(1+\alpha) - (\alpha + \beta)) \frac{(a+1)n(b+1)n}{(c+1)n(1)n+1}$$

$$= (1+\alpha) \sum_{n=0}^{\infty} (n+1) \frac{(a+1)n(b+1)n}{(c+1)n(1)n} + (2+\alpha - \beta) \sum_{n=0}^{\infty} \frac{c}{(c+1)n(1)n}$$

$$+ (1-\beta) \sum_{n=0}^{\infty} \frac{(a+1)n(b+1)n}{(c+1)n(1)n+1}$$

$$= \frac{(1+\alpha)(a+1)(b+1)}{c+1} \sum_{n=0}^{\infty} \frac{(a+2)n(b+2)n}{(c+2)n(1)n} + (3+2\alpha - \beta) \sum_{n=0}^{\infty} \frac{(a+1)n(b+1)n}{(c+1)n(1)n}$$

$$+ (1-\beta) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)n(b)n}{(c)n(1)n}$$

$$= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \left( (1+\alpha)(a+1)(b+1) + (3+2\alpha - \beta)(c-a-b-2) \right) - \frac{(1-\beta)c}{ab}.$$

This last expression is bounded above by $(1-\beta)c/|ab|$ if and only if

$$(1+\alpha)(a+1)(b+1) + (3+2\alpha - \beta)(c-a-b-2) + \frac{1-\beta}{ab} (c-a-b-2) \leq 0, \quad (2.18)$$

which is equivalent to (2.14).

(ii) In view of (ii) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} n(n+\alpha) - (\alpha + \beta) \frac{(a)\beta n-1(b)\beta n-1}{(c)\beta n-1(1)\beta n-1} \leq 1-\beta. \quad (2.19)$$

Now,

$$\sum_{n=0}^{\infty} (n+2)((n+2)(1+\alpha) - (\alpha + \beta)) \frac{(a)\beta n+1(b)\beta n+1}{(c)\beta n+1(1)\beta n+1}$$

$$= (1+\alpha) \sum_{n=0}^{\infty} (n+2)^2 \frac{(a)\beta n+1(b)\beta n+1}{(c)\beta n+1(1)\beta n+1} - (\alpha + \beta) \sum_{n=0}^{\infty} (n+2) \frac{(a)\beta n+1(b)\beta n+1}{(c)\beta n+1(1)\beta n+1}. \quad (2.20)$$
Writing \( n + 2 = (n + 1) + 1 \), we have

\[
\sum_{n=0}^{\infty} (n+2)(a)_{n+1}(b)_{n+1} = \sum_{n=0}^{\infty} (a)_{n+1}(b)_{n+1} + \sum_{n=0}^{\infty} (a)_{n+1}(b)_{n+1},
\]

\[
\sum_{n=0}^{\infty} (n+2)2(a)_{n+1}(b)_{n+1} = \sum_{n=0}^{\infty} (n+1)(a)_{n+1}(b)_{n+1} + 2\sum_{n=0}^{\infty} (a)_{n+1}(b)_{n+1} + \sum_{n=0}^{\infty} (a)_{n+1}(b)_{n+1}
\]

\[
= \sum_{n=1}^{\infty} (a)_{n+1}(b)_{n+1} + 3\sum_{n=0}^{\infty} (a)_{n+1}(b)_{n+1} + \sum_{n=1}^{\infty} (a)_{n+1}(b)_{n+1}.
\]  

(2.21)

Substituting (2.21) into the right-hand side of (2.20), we obtain

\[
(1 + \alpha)\sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{2n+2}}{(c)_{n+2}} + (3 + 2\alpha - \beta)\sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}} + (1 - \beta)\sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}.
\]

(2.22)

Since \((a)_{n+k} = (a)_k(a+k)_n\), we write (2.22) as

\[
(1 + \alpha)(a)_{2}(b)_2 \frac{\Gamma(c + 2)\Gamma(c - a - b - 2)}{\Gamma(c - a)\Gamma(c - b)} + (3 + 2\alpha - \beta) \frac{ab\Gamma(c + 1)\Gamma(c - a - b - 1)}{c\Gamma(c - a)\Gamma(c - b)}
\]

\[
+ (1 - \beta) \left( \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} - 1 \right).
\]

(2.23)

By simplification, we see that the last expression is bounded above by \(1 - \beta\) if and only if (2.15) holds. \(\square\)

**Theorem 2.5.** (i) If \(a, b > -1, c > 0,\) and \(ab < 0,\) then \(z F(a, b; c; z)\) is in \(PT(a)\) if and only if

\[
c \geq a + b + 1 - \frac{ab}{a}.
\]

(2.24)

(ii) If \(a, b > 0\) and \(c > a + b + 1,\) then \(F_1(a, b; c; z) = z(2 - F(a, b; c; z))\) is in \(PT(a)\) if and only if

\[
\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left( 1 + \frac{ab}{a(c - a - b - 1)} \right) \leq 2.
\]

(2.25)

**Proof.** (i) Since

\[
z F(a, b; c; z) = z + \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a + 1)_{n-2}(b + 1)_{n-2}}{(c + 1)_{n-2}(1)_{n-1}} z^n = \sum_{n=2}^{\infty} \frac{(a + 1)_{n-2}(b + 1)_{n-2}}{(c + 1)_{n-2}(1)_{n-1}} z^n,
\]

(2.26)

according to (i) of Lemma 2.2, we must show that

\[
\sum_{n=2}^{\infty} (n-1 + \alpha) \frac{(a + 1)_{n-2}(b + 1)_{n-2}}{(c + 1)_{n-2}(1)_{n-1}} \leq \frac{c}{|ab|} a.
\]

(2.27)
Noting that \((\lambda)_n = \lambda(\lambda + 1)_{n-1}\) and then applying (1.6), we have

\[
\sum_{n=0}^{\infty} (n + 1 + a) \frac{(a + 1)n(b + 1)n}{(c + 1)n(1)n+1} = \sum_{n=0}^{\infty} \frac{(a + 1)n(b + 1)n}{(c + 1)n(1)n} + \alpha \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)n(b)n}{(c)n(1)n}
\]

\[
= \frac{\Gamma(c + 1)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)} + \alpha \frac{c}{ab} \left( \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} - 1 \right).
\]

(2.28)

Hence, (2.27) is equivalent to

\[
\frac{\Gamma(c + 1)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)} \left( 1 + \alpha \frac{c - a - b - 1}{ab} \right) \leq \alpha \left( \frac{c}{|ab|} - \frac{c}{ab} \right) = 0.
\]

(2.29)

Thus, (2.29) is valid if and only if \(1 + \alpha(c - a - b - 1)/ab \leq 0\) or, equivalently, \(c \geq a + b + 1 - ab/\alpha\).

(ii) Since

\[
F_1(a, b; c; z) = z - \sum_{n=1}^{\infty} \frac{(a)(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n,
\]

by (i) of Lemma 2.2, we need only to show that

\[
\sum_{n=2}^{\infty} (n - 1 + a) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq \alpha.
\]

(2.31)

Now,

\[
\sum_{n=2}^{\infty} (n - 1 + a) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \sum_{n=1}^{\infty} \frac{n(a)n(b)n}{(c)n(1)n} + \alpha \sum_{n=1}^{\infty} \frac{(a)n(b)n}{(c)n(1)n}
\]

\[
= \frac{ab}{c} \sum_{n=1}^{\infty} \frac{(a + 1)n-1(b + 1)n-1}{(c + 1)n-1(1)n-1} + \alpha \sum_{n=1}^{\infty} \frac{(a)n(b)n}{(c)n(1)n}
\]

\[
= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left( \frac{ab}{c - a - b - 1} + \alpha \right) - \alpha.
\]

(2.32)

But this last expression is bounded above by \(\alpha\) if and only if (2.25) holds.

\[\square\]

**Theorem 2.6.** (i) If \(a, b > -1, \ ab < 0, \text{and} \ c > a + b + 2, \text{then} \ zF(a, b; c; z) \text{is in} \ C_\mathcal{D}_\mathcal{T}(\alpha) \text{if and only if}

\[
(a)_{2}^{2} + (2 + a)ab(c - a - b - 2) + a(c - a - b - 2)_{2} \geq 0.
\]

(2.33)

(ii) If \(a, b > 0 \text{and} \ c > a + b + 2, \text{then} \ F_1(a, b; c; z) = z(2 - F(a, b; c; z)) \text{is in} \ C_\mathcal{D}_\mathcal{T}(\alpha) \text{if and only if}

\[
\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left( \frac{(a)_{2}^{2}}{\alpha(c - a - b - 2)_{2}} + \left( \frac{2 + a}{\alpha} \right) \left( \frac{ab}{c - a - b - 1} \right) + 1 \right) \leq 2.
\]

(2.34)
Proof. (i) Since $z\bar{F}$ has the form (2.26), we see from (ii) of Lemma 2.2 that our conclusion is equivalent to

$$
\sum_{n=2}^{\infty} n(n-1+\alpha) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \frac{c}{|ab|} \alpha.
$$

(2.35)

Writing $(n+2)(n+1+\alpha) = (n+1)^2 + (1+\alpha)(n+1) + \alpha$, we see that

$$
\sum_{n=0}^{\infty} (n+2)(n+1+\alpha) \frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n+1}}
$$

$$
= \sum_{n=0}^{\infty} (n+1) \frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n}} + (1+\alpha) \sum_{n=0}^{\infty} (a+1)_{n}(b+1)_{n} + a \sum_{n=0}^{\infty} (c+1)_{n}(1)_{n+1}
$$

$$
= \frac{(a+1)(b+1)}{c+1} \sum_{n=0}^{\infty} \frac{(a+2)_{n}(b+2)_{n}}{(c+2)_{n}(1)_{n}} + (2+\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n}} + \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}
$$

$$
= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \left( (a+1)(b+1) + (2+\alpha)(c-a-b-2) + \frac{\alpha}{ab} (c-a-b)^2 \right) - \frac{ac}{ab}.
$$

(2.36)

This last expression is bounded above by $ac/|ab|$ if and only if $(a+1)(b+1) + (2+\alpha)(c-a-b-2) + (\alpha/ab)(c-a-b)^2 \leq 0$, which is equivalent to (2.33).

(ii) In view of (ii) of Lemma 2.2, we need only to show that

$$
\sum_{n=2}^{\infty} n(n-1+\alpha) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq \alpha.
$$

(2.37)

Now,

$$
\sum_{n=0}^{\infty} (n+2)(n+2-(1-\alpha)) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} = \sum_{n=0}^{\infty} (n+2) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} - (1-\alpha) \sum_{n=0}^{\infty} (n+2) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}.
$$

(2.38)

Substituting (2.21) into the right-hand side of (2.38), we obtain

$$
\sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n+1}} + (2+\alpha) \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + a \sum_{n=1}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.
$$

(2.39)

Since $(a)_{n+k} = (a)_{k}(a+k)_{n}$, we may write (2.39) as

$$
\frac{(a)_{2}(b)_{2}}{(c)_{2}} \frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} + (2+\alpha) \frac{ab\Gamma(c+1)\Gamma(c-a-b-1)}{c\Gamma(c-a)\Gamma(c-b)} + \alpha \left( \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right).
$$

(2.40)

By simplification, we see that the last expression is bounded above by $\alpha$ if and only if (2.34) holds. \qed
3. An integral operator

In the next theorems, we obtain similar-type results in connection with a particular integral operator \( G(a, b; c; z) \) acting on \( F(a, b; c; z) \) as follows:

\[
G(a, b; c; z) = \int_0^z F(a, b; c; t) dt.
\]  

(3.1)

**Theorem 3.1.** Let \( a, b > -1, ab < 0 \), and \( c > \max\{0, a + b\} \). Then,

(i) \( G(a, b; c; z) \) defined by (3.1) is in \( S_{P\mathcal{T}}(\alpha, \beta) \) if and only if

\[
\frac{\Gamma(c + 1)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left( \frac{1}{ab} - \frac{(a + \beta)(c - a - b)}{(a - 1)z(b - 1)_2} \right) + \frac{(a + \beta)(c - 1)_2}{(a - 1)z(b - 1)_2} \leq 0;
\]

(3.2)

(ii) \( G(a, b; c; z) \) defined by (3.1) is in \( P\mathcal{T}(\alpha) \) if and only if

\[
\frac{\Gamma(c + 1)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left( \frac{1}{ab} + \frac{(a - 1)(c - a - b)}{(a - 1)z(b - 1)_2} \right) - \frac{(a - 1)(c - a)_2}{(a - 1)z(b - 1)_2} \leq 0.
\]

(3.3)

**Proof.** (i) Since

\[
G(a, b; c; z) = z - \frac{|ab|}{c} \sum_{n=2}^{\infty} \frac{(a + 1)_{n-2}(b + 1)_{n-2}}{(c + 1)_{n-2}(1)_n} z^n,
\]

(3.4)

by (i) of Lemma 2.1, we need only to show that

\[
\sum_{n=2}^{\infty} \frac{(n(1 + \alpha) - (a + \beta))}{(c + 1)_{n-2}(1)_n} \leq (1 - \beta) \frac{c}{|ab|}.
\]

(3.5)

Now,

\[
\sum_{n=0}^{\infty} \frac{(n + 2)(1 + \alpha) - (a + \beta)}{(c + 1)_{n+1}(1)_{n+2}} \frac{(a + 1)_{n}(b + 1)_n}{(c + 1)_{n}(1)_n} + \frac{\Gamma(c + 1)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left( \frac{1}{ab} - \frac{(a + \beta)(c - a - b)}{(a - 1)z(b - 1)_2} \right)
\]

\[
\leq (1 - \beta) \frac{c}{|ab|},
\]

(3.6)

which is equivalent to (3.2).
(ii) According to (i) of Lemma 2.2, it is sufficient to show that
\[ \sum_{n=0}^{\infty} (n + 1 + \alpha) \frac{(a + 1)n(b + 1)n}{(c + 1)n^2(1)_n}, \]
which is equivalent to
\[ \sum_{n=0}^{\infty} (n + 1 + \alpha) \frac{(a + 1)n(b + 1)n}{(c + 1)n^2(1)_n} \leq \frac{\alpha c}{|ab|}. \] (3.7)

Now,
\[
\sum_{n=0}^{\infty} (n + 1 + \alpha) \frac{(a + 1)n(b + 1)n}{(c + 1)n^2(1)_n+2}
\]
\[
= \sum_{n=0}^{\infty} \frac{(a + 1)n(b + 1)n}{(c + 1)n^2(1)_n+1} + (\alpha - 1) \sum_{n=0}^{\infty} \frac{(a + 1)n(b + 1)n}{(c + 1)n^2(1)_n+2}
\]
\[
= \frac{c}{ab} \left( \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b) - 1} \right)
\]
\[
+ (\alpha - 1) \frac{c}{ab} \left( \frac{(c - 1)}{(a - 1)(b - 1)} \left( \frac{\Gamma(c - 1)\Gamma(c - a - b + 1)}{\Gamma(c - a)\Gamma(c - b)} - 1 \right) \right)
\]
\[
= \frac{\Gamma(c + 1)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left( \frac{1}{ab} + \frac{(a - 1)(c - a - b)}{(a - 1)(b - 1)^2} - \frac{(a - 1)(c - 1)}{(a - 1)(b - 1)^2} - \frac{\alpha c}{ab} \right)
\]
\[
\leq \frac{\alpha c}{|ab|},
\]
which is equivalent to (3.3).

Now, we observe that \( G(a, b; c; z) \in \mathcal{MC}(\alpha, \beta)(\mathcal{CP}(\alpha)) \) if and only if \( zF(a, b; c; z) \in S_p(\alpha, \beta)(\mathcal{CP}(\alpha)) \). Thus, any result of functions belonging to the class \( S_p(\alpha, \beta)(\mathcal{CP}(\alpha)) \) about \( zF \) leads to that of functions belonging to the class \( \mathcal{MC}(\alpha, \beta)(\mathcal{CP}(\alpha)) \). Hence, we obtain the following analogues to Theorems 2.3 and 2.5.

**Theorem 3.2.** Let \( a, b > -1, \ ab < 0, \text{ and } c > a + b + 2. \) Then,

(i) \( G(a, b; c; z) \) defined by (3.1) is in \( \mathcal{MC}(\alpha, \beta) \) if and only if
\[
c \geq a + b + 1 - \frac{(1 + a)ab}{(1 - \beta)}, \] (3.9)

(ii) \( G(a, b; c; z) \) defined by (3.1) is in \( \mathcal{CP}(\alpha) \) if and only if
\[
c \geq a + b + 1 - \frac{ab}{\alpha}. \] (3.10)

**References**


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<thead>
<tr>
<th>Manuscript Due</th>
<th>March 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>June 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>September 1, 2009</td>
</tr>
</tbody>
</table>

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