Research Article

Elliptic Equations in Weighted Sobolev Spaces on Unbounded Domains

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We study in this paper a class of second-order linear elliptic equations in weighted Sobolev spaces on unbounded domains of \( \mathbb{R}^n, n \geq 3 \). We obtain an a priori bound, and a regularity result from which we deduce a uniqueness theorem.

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1. Introduction

Let \( \Omega \) be an open subset of \( \mathbb{R}^n, n \geq 3 \). Assign in \( \Omega \) the uniformly elliptic second-order linear differential operator

\[
L = - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a,
\]

(1.1)

with coefficients \( a_{ij} = a_{ji} \in L^\infty(\Omega), i, j = 1, \ldots, n \), and consider the associate Dirichlet problem:

\[
\begin{align*}
  u & \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \\
  Lu &= f, \quad f \in L^p(\Omega),
\end{align*}
\]

(1.2)

where \( p \in [1, +\infty[ \).

It is well known that if \( \Omega \) is a bounded and sufficiently regular set, the above problem has been widely investigated by several authors under various hypotheses of discontinuity on the leading coefficients, in the case \( p = 2 \) or \( p \) sufficiently close to 2. In particular, some \( W^{2,p} \)-bounds for the solutions of the problem (1.2) and related existence and uniqueness results have been obtained. Among the other results on this subject, we quote here those
proved in [1], where the author assumed that \( a_{ij} \)’s belong to \( W^{1,n}(\Omega) \) (and considered the case \( p = 2 \)) and in [2–4] (where the coefficients belong to some classes wider than \( W^{1,n}(\Omega) \)). More recently, a relevant contribution has been given in [5–8], where the coefficients \( a_{ij} \) are assumed to be in the class VMO and \( p \in ]1, +\infty[ \); observe here that VMO contains the space \( W^{1,n}(\Omega) \).

If the set \( \Omega \) is unbounded and regular enough, under assumptions similar to those required in [1], problem (1.2) has for instance been studied in [9–11] with \( p = 2 \), and in [12] with \( p \in ]1, +\infty[ \). Instead, in [13, 14] the leading coefficients satisfy restrictions similar to those in [5, 6].

In this paper, we extend some results of [13, 14] to a weighted case. More precisely, we denote by \( \rho \) a weight function belonging to a suitable class such that

\[
\inf_{\Omega} \rho > 0, \quad \lim_{|x| \to +\infty} \rho(x) = +\infty, \tag{1.3}
\]

and consider the Dirichlet problem:

\[
\begin{align*}
u & \in W^{2,p}_s(\Omega) \cap W^{1,p}_s(\Omega), \\
Lu & = f, \quad f \in L^p(\Omega),
\end{align*}
\tag{1.4}
\]

where \( s \in \mathbb{R}, W^{2,p}_s(\Omega), W^{1,p}_s(\Omega) \), and \( L^p(\Omega) \) are some weighted Sobolev spaces and the weight functions are a suitable power of \( \rho \). We obtain an a priori bound for the solutions of (1.4). Moreover, we state a regularity result that allows us to deduce a uniqueness theorem for the problem (1.4). A similar weighted case was studied in [15] with the leading coefficients satisfying hypotheses of Miranda’s type and when \( p = 2 \).

2. Weight functions and weighted spaces

Let \( G \) be any Lebesgue measurable subset of \( \mathbb{R}^n \) and let \( \Sigma(G) \) be the collection of all Lebesgue measurable subsets of \( G \). If \( F \in \Sigma(G) \), denote by \( |F| \) the Lebesgue measure of \( F \), by \( \chi_F \) the characteristic function of \( F \), by \( F(x,r) \) the intersection \( F \cap B(x,r) \) (\( x \in \mathbb{R}^n, r \in \mathbb{R}_+ \))—where \( B(x,r) \) is the open ball of radius \( r \) centered at \( x \)—and by \( \mathcal{S}(F) \) the class of restrictions to \( F \) of functions \( \zeta \in C_0^\infty(\mathbb{R}^n) \) with \( \overline{F} \cap \text{supp} \zeta \subseteq F \). Moreover, if \( X(F) \) is a space of functions defined on \( F \), we denote by \( X_{\text{loc}}(F) \) the class of all functions \( g : F \to \mathbb{R} \), such that \( \zeta g \in X(F) \) for any \( \zeta \in \mathcal{S}(F) \).

We introduce a class of weight functions defined on an open subset \( \Omega \) of \( \mathbb{R}^n \). Denote by \( \mathcal{A}(\Omega) \) the set of all measurable functions \( \rho : \Omega \to \mathbb{R}_+ \), such that

\[
\gamma^{-1} \rho(y) \leq \rho(x) \leq \gamma \rho(y) \quad \forall y \in \Omega, \quad \forall x \in \Omega(y, \rho(y)),
\tag{2.1}
\]

where \( \gamma \in \mathbb{R}_+ \) is independent of \( x \) and \( y \). Examples of functions in \( \mathcal{A}(\Omega) \) are the function

\[
x \in \mathbb{R}^n \mapsto 1 + a|x|, \quad a \in ]0, 1[,
\tag{2.2}
\]

and, if \( \Omega \neq \mathbb{R}^n \) and \( S \) is a nonempty subset of \( \partial \Omega \), the function

\[
x \in \Omega \mapsto a \text{dist}(x, S), \quad a \in ]0, 1[.
\tag{2.3}
\]
For $\rho \in \mathcal{A}(\Omega)$, we put

$$S_\rho = \{ z \in \partial \Omega : \lim_{x \to z} \rho(x) = 0 \}. \quad (2.4)$$

It is known that

$$\rho \in L^\infty_{\text{loc}}(\Omega), \quad \rho^{-1} \in L^\infty_{\text{loc}}(\overline{\Omega} \setminus S_\rho) \quad (2.5)$$

(see [16, 17]).

We assign an unbounded open subset $\Omega$ of $\mathbb{R}^n$.

Let $\rho_1$ be a function, such that $\rho_1 \in \mathcal{A}(\mathbb{R}^n)$ and

$$\inf_\Omega \rho_1 > 0, \quad \lim_{|x| \to +\infty} \rho_1(x) = +\infty. \quad (2.6)$$

We put

$$\rho = \rho_1|_{\Omega}. \quad (2.7)$$

For any $a \in [0, 1]$ and $x \in \mathbb{R}^n$, we set

$$I_a(x) = \Omega(x, a\rho_1(x)). \quad (2.8)$$

If $k \in \mathbb{N}_0$, $1 \leq p < +\infty$, $s \in \mathbb{R}$, and $\rho \in \mathcal{A}(\Omega)$, consider the space $W^{k,p}_s(\Omega)$ of distributions $u$ on $\Omega$, such that $\rho^s \partial^a u \in L^p(\Omega)$ for $|a| \leq k$, equipped with the norm

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|a| \leq k} \|\rho^s \partial^a u\|_{L^p(\Omega)}. \quad (2.9)$$

Moreover, denote by $W^{k,p}_s(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}_s(\Omega)$ and put $W^{k,p}_s(\Omega) = L^p_0(\Omega)$. A more detailed account of properties of the above defined spaces can be found, for instance, in [18].

From [15, Lemmas 1.1 and 2.1], we deduce the following two lemmas, respectively.

**Lemma 2.1.** For any $p \in [1, +\infty[, s \in \mathbb{R}$, and $a \in [0, 1]$, $g \in L^p_0(\Omega)$ if and only if $g \in L^p_0(\overline{\Omega})$ and the function $x \in \mathbb{R}^n \rightarrow \rho_1^{s-n/p}(x)\|g\|_{L^p(I_a(x))}$ belongs to $L^p(\mathbb{R}^n)$. Moreover, there exist $c_1, c_2 \in \mathbb{R}_+$, such that

$$c_1 \|g\|_{L^p_0(\Omega)} \leq \int_{\mathbb{R}^n} \rho_1^{s-n/p}(x)\|g\|_{L^p(I_a(x))}^p \, dx \leq c_2 \|g\|_{L^p_0(\Omega)}^p \quad \forall g \in L^p_0(\Omega), \quad (2.10)$$

where $c_1$ and $c_2$ depend on $n, p, s, a$, and $\rho$.

**Lemma 2.2.** If $\Omega$ has the segment property, then for any $k \in \mathbb{N}_0$, $p \in [1, +\infty[, \rho \in \mathcal{A}(\mathbb{R}^n)$, and $s \in \mathbb{R}$ one has

$$W^{k,p}_s(\Omega) \cap W^{k,p}_s(\overline{\Omega}) = W^{k,p}_s(\Omega). \quad (2.11)$$
3. Some embedding lemmas

We now recall the definitions of the function spaces in which the coefficients of the operator will be chosen. If $\Omega$ has the property

$$|\Omega(x, r)| \geq Ar^n \quad \forall x \in \Omega, \forall r \in ]0, 1],$$

(3.1)

where $A$ is a positive constant independent of $x$ and $r$, it is possible to consider the space $\text{BMO}(\Omega, \tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^1_{\text{loc}}(\overline{\Omega})$ such that

$$[g]_{\text{BMO}(\Omega, \tau)} = \sup_{x \in \Omega} \frac{1}{\tau} \sup_{r \in [0, \tau]} \left| \frac{r^n}{|\Omega(x, r)|} \int_{\Omega(x, r)} g - \int_{\Omega} g \right| < +\infty,$$

(3.2)

where

$$\int_{\Omega(x, r)} g = |\Omega(x, r)|^{-1} \int_{\Omega} g.$$

(3.3)

If $g \in \text{BMO}(\Omega) = \text{BMO}(\Omega, \tau_A)$, where

$$\tau_A = \sup \left\{ \tau \in \mathbb{R}_+ : \sup_{x \in \Omega} \frac{1}{\tau} \sup_{r \in [0, \tau]} \left| \frac{r^n}{|\Omega(x, r)|} \right| \leq \frac{1}{A} \right\},$$

(3.4)

we will say that $g \in \text{VMO}(\Omega)$ if $[g]_{\text{BMO}(\Omega, \tau)} \to 0$ for $\tau \to 0^+$. A function $\eta[g] : ]0, 1] \to \mathbb{R}_+$ is called a modulus of continuity of $g$ in $\text{VMO}(\Omega)$ if

$$[g]_{\text{BMO}(\Omega)} \leq \eta[g](\tau) \quad \forall \tau \in ]0, 1], \quad \lim_{\tau \to 0^+} \eta[g](\tau) = 0.$$

(3.5)

For $t \in [1, +\infty[$ and $\lambda \in [0, n]$, we denote by $M^{t, 1}(\Omega)$ the set of all functions $g$ in $L^1_{\text{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^{t, 1}(\Omega)} = \sup_{r \in [0, 1]} r^{-1/t} \|g\|_{L^t(\Omega(x, r))} < +\infty,$$

(3.6)

endowed with the norm defined by (3.6). Then, we define $M_0^{t, 1}(\Omega)$ as the closure of $C^\infty(\Omega)$ in $M^{t, 1}(\Omega)$. In particular, we put $M^t(\Omega) = M^{1,0}(\Omega)$, and $M_0^t(\Omega) = M_0^{1,0}(\Omega)$. In order to define the modulus of continuity of a function $g$ in $M_0^{t, 1}(\Omega)$, recall first that for a function $g \in M^{t, 1}(\Omega)$ the following characterization holds:

$$g \in M_0^{t, 1}(\Omega) \iff \lim_{\tau \to 0^+} (p_g(\tau) + \|(1 - \zeta_{1/\tau})g\|_{M^{t, 1}(\Omega)}) = 0,$$

(3.7)

where

$$p_g(\tau) = \sup_{E \in \Sigma(\Omega)} \sup_{\sup_{\text{dist}(F(x, 1)) \leq \tau} E(x, 1) \leq \tau} \|\chi_E g\|_{M^{t, 1}(\Omega)},$$

(3.8)
and $\zeta_r, r \in \mathbb{R}_+$, is a function in $C_0^\infty(\mathbb{R}^n)$ such that

$$0 \leq \zeta_r \leq 1, \quad \zeta_r|_{\mathbb{R}_+^n} = 1, \quad \text{supp} \ zeta_r \subset B_{2r},$$

(3.9)

with the position $B_r = B(0, r)$. Thus, the modulus of continuity of $g \in M_0^1(\Omega)$ is a function

$$\sigma_c[g] : [0, 1] \rightarrow \mathbb{R}_+,$$

such that

$$p_\varepsilon(\tau) + \| (1 - \zeta_1/r) g \|_{M^1(\Omega)} \leq \sigma_c[g](\tau) \quad \forall \tau \in [0, 1], \quad \lim_{\tau \rightarrow 0} \sigma_c[g](\tau) = 0.$$  

(3.11)

A more detailed account of properties of the above defined function spaces can be found in [9, 19, 20].

We consider the following condition:

(h0) $\Omega$ has the cone property, $p \in [1, +\infty[, s \in \mathbb{R}, k, h, t$ are numbers such that

$$k \in \mathbb{N}, \quad h \in \{0, 1, \ldots, k-1\}, \quad t \geq p, \quad t > p \quad \text{if } p = \frac{n}{k-h}, \quad g \in M^1(\Omega).$$

(3.12)

From [21, Theorem 3.1] we have the following.

**Lemma 3.1.** If the assumption (h0) holds, then for any $u \in W^{k,p}_s(\Omega)$ one has $g \partial^h u \in L^p(\Omega)$ and

$$\| g \partial^h u \|_{L^p(\Omega)} \leq c \| g \|_{M^1(\Omega)} \| u \|_{W^{k,p}_s(\Omega)},$$

(3.13)

with $c$ dependent only on $\Omega, n, k, h, p, t, s$.

From [21, Theorem 3.2] it follows Lemma 3.2.

**Lemma 3.2.** If the assumption (h0) is satisfied and in addition $g \in M^1_s(\Omega)$, then for any $\varepsilon \in \mathbb{R}_+$ there exist a constant $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open set $\Omega_e \subset \subset \Omega$, with the cone property, such that

$$\| g \partial^h u \|_{L^p(\Omega)} \leq \varepsilon \| u \|_{W^{k,p}_s(\Omega)} + c(\varepsilon) \| u \|_{L^p(\Omega)}, \quad \forall u \in W^{k,p}_s(\Omega),$$

(3.14)

where $c(\varepsilon), \Omega_e$ depend on $\varepsilon, \Omega, n, k, h, p, t, s, k, s$, and $\sigma_c[g]$.

**4. An a priori bound**

Assume that $\Omega$ is an unbounded open subset of $\mathbb{R}^n, n \geq 3$, with the uniform $C^{1,1}$-regularity property, and let $\rho$ be the function defined by (2.7). Moreover, let $p \in [1, +\infty[ \quad \text{and} \quad s \in \mathbb{R}$.

Consider in $\Omega$ the differential operator:

$$L = -\sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a,$$

(4.1)
Theorem 4.1. If the hypotheses \( \Omega \) and a bounded open subset \( \Omega \) with the following conditions on the coefficients:

\[
\begin{align*}
(a) & \quad a_{ij} = a_{ji} \in L^\infty(\Omega) \cap \text{VMO}_{\text{loc}}(\Omega), \quad i, j = 1, \ldots, n, \\
(b) & \quad \exists \nu > 0 : \sum_{i,j=1}^n a_{ij} h_i^2 \geq \nu |h|^2 \text{ a.e. in } \Omega, \quad \forall h \in \mathbb{R}^n,
\end{align*}
\]

(4.2)

there exist functions \( e_{ij}, i, j = 1, \ldots, n, \) and \( \mu \in \mathbb{R}_+ \) such that

\[
\begin{align*}
(e) & \quad e_{ij} \in L^\infty(\Omega), \quad (e_{ij})_{x_n} \in M_c^1(\Omega), \quad \text{with } t \in [2, n], \ i, j, h = 1, \ldots, n, \\
& \quad \sum_{i,j=1}^n e_{ij} h_i^2 \geq \mu |h|^2 \text{ a.e. in } \Omega, \quad \forall h \in \mathbb{R}^n, \\
& \quad g \in L^\infty(\Omega), \quad g_0 = \text{ess inf}_\Omega g > 0, \\
& \quad \lim_{r \to 0} \sum_{i,j=1}^n \|e_{ij} - ga_{ij}\|_{L^\infty(\Omega, B_r)} = 0,
\end{align*}
\]

(4.3)

\[
\begin{align*}
(f) & \quad a_i \in M^{1\alpha}(\Omega), \quad i = 1, \ldots, n, \\
& \quad a = a' + a'', \quad a' \in M^{1\alpha}(\Omega), \quad a'' \in L^\infty(\Omega), \quad \text{ess inf}_\Omega a' = a'' > 0,
\end{align*}
\]

(4.4)

where

\[
\begin{align*}
t_1 & \geq n \quad \text{if } p < n, \quad t_1 > n \quad \text{if } p = n, \quad t_1 = p \quad \text{if } p > n, \\
t_2 & \geq n/2 \quad \text{if } p < n/2, \quad t_2 > n/2 \quad \text{if } p = n/2, \quad t_2 = p \quad \text{if } p > n/2.
\end{align*}
\]

(4.5)

Observe that under the assumptions (h1)-(h3), it follows that the operator \( L : W_2^p(\Omega) \to L_2^p(\Omega) \) is bounded from Lemma 3.1.

Theorem 4.1. If the hypotheses (h1), (h2), and (h3) are verified, then there exist a constant \( c \in \mathbb{R}_+ \) and a bounded open subset \( \Omega_0 \subset \Omega \), with the cone property, such that

\[
\|u\|_{W_2^{2p}(\Omega)} \leq c\left(\|Lu\|_{L_2^p(\Omega)} + \|u\|_{L_2^p(\Omega_0)}\right), \quad \forall u \in W_2^{2p}(\Omega) \cap W_2^{1,p}(\Omega),
\]

(4.6)

with \( c \) and \( \Omega_0 \) depending on \( n, p, \rho, s, \Omega, \nu, \mu, g_0, a''_0, t, t_1, t_2, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \|\eta (e_{ij} x_n)\|, \sigma_s(a_{ij}), \sigma_s(a'_{ij}), \sigma_s(a'_{ij}) \), where \( r_0 \in \mathbb{R}_+ \) depends on \( n, p, \Omega, \mu, g_0, a''_0, t, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a''\|_{L^\infty(\Omega)}, \sigma_s(\{e_{ij}\} x_n) \).

Proof. We consider a function \( \phi \in C_0^\infty(\mathbb{R}^n) \), such that

\[
\begin{align*}
\phi|_{B_1} & = 1, \quad \supp \phi \subset B_1, \\
\sup_{\mathbb{R}^n} |\partial^\alpha \phi| & \leq c_{\alpha} \quad \forall \alpha \in \mathbb{N}_0^n.
\end{align*}
\]

(4.7)
where \( c_\alpha \in \mathbb{R}_+ \) depends only on \( \alpha \), fix \( y \in \mathbb{R}^n \) and put
\[
q_r = q_{r,y} : x \in \mathbb{R}^n \longrightarrow \phi \left( \frac{x - y}{\rho_1(y)} \right).
\]

Clearly we have
\[
q_{B(y,(1/2)\rho_1(y))} = 1, \quad \text{supp } q \subset B(y, \rho_1(y)),
\]
\[
\sup_{\mathbb{R}^n} | \partial^\alpha q | \leq c_\alpha \rho_1^{-|\alpha|}(y) \quad \forall \alpha \in \mathbb{N}_0^n.
\]

Now, we put
\[
L_0 = -\sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}
\]
and fix \( u \in W^{2,p}_s(\Omega) \cap W^{1,p}_0(\Omega) \). Since \( q_r u \in W^{2,p}(\Omega) \cap W^{1,p}_s(\Omega) \), from [14, Theorem 3.3], it follows that there exist \( c_1 \in \mathbb{R}_+ \) and a bounded open subset \( \Omega_1 \subset \subset \Omega \), with the cone property, such that
\[
\| q_r u \|_{W^{2,p}(\Omega)} \leq c_1 (\| (L_0 + a'') (q_r u) \|_{L^p(\Omega)} + \| q_r u \|_{L^p(\Omega_1)}),
\]
with \( c_1 \) and \( \Omega_1 \) depending on \( n, p, \Omega, \nu, \mu, \varrho_0, a_0', t, \| a_{ij} \|_{L^\infty(\Omega)}, \| e_{ij} \|_{L^\infty(\Omega)}, \| g \|_{L^\infty(\Omega)}, \| a'' \|_{L^\infty(\Omega)}, \eta [\xi_{2a}, a_{ij} \sigma_0 (e_{ij})_x], \sigma_0 (e_{ij})_x \), where \( r_0 \in \mathbb{R}_+ \) depends on \( n, p, \Omega, \nu, \mu, \varrho_0, a_0', t, \| e_{ij} \|_{L^\infty(\Omega)}, \| g \|_{L^\infty(\Omega)}, \| a'' \|_{L^\infty(\Omega)}, \| a'' \|_{L^\infty(\Omega)}, \sigma_0 (e_{ij})_x \). Since
\[
L_0 (q_r u) = q_r L_0 u - 2 \sum_{i,j=1}^n a_{ij} q_r x_i u_{x_j} - \sum_{i,j=1}^n a_{ij} q_r x_i u_x,
\]
from (4.11) and (4.12), we have
\[
\| q_r u \|_{W^{2,p}(\Omega)}
\]
\[
\leq c_2 \left( \| q_r (L_0 + a'') u \|_{L^p(\Omega)} + \sum_{i,j=1}^n \| q_r x_i u_{x_j} \|_{L^p(\Omega)} + \sum_{i,j=1}^n \| q_r x_i u_x \|_{L^p(\Omega)} + \| q_r u \|_{L^p(\Omega_1)} \right),
\]
with \( c_2 \) dependent on the same parameters of \( c_1 \).

On the other hand, since \( \rho \in L^\infty_{\text{loc}} (\overline{\Omega}) \), we have that
\[
\| q_r u \|_{L^p(\Omega_1)} \leq c_3 \rho_1^{-2} (y) \| u \|_{L^p(I_1(y))},
\]
where \( c_3 \in \mathbb{R}_+ \) depends only on \( \rho \).

Therefore, by (4.13) and (4.14), we deduce the bound:
\[
\| u \|_{W^{2,p}(I_1(y))} \leq \| q_r u \|_{W^{2,p}(\Omega)}
\]
\[
\leq c_4 (\| L_0 u + a'' u \|_{L^p(I_1(y))} + \rho_1^{-1} (y) \| u_x \|_{L^p(I_1(y))} + \rho_1^{-2} (y) \| u \|_{L^p(I_1(y))}),
\]
where \( c_4 \in \mathbb{R}_+ \) depends on the same parameters of \( c_2 \) and on \( \rho \).
From (4.15) it follows
\[
\int_{\mathbb{R}^n} \rho_1^{p_{s-n}}(y) \|u\|_{W^{2,p}(I_t(y))}^p \, dy \\
\leq c_5 \left( \int_{\mathbb{R}^n} \rho_1^{p_{s-n}}(y) \|L_0 u + a'u\|_{L^p(I_t(y))}^p \, dy \\
+ \int_{\mathbb{R}^n} \rho_1^{p_{s-n}-p}(y) \|u_x\|_{L^p(I_t(y))}^p \, dy + \int_{\mathbb{R}^n} \rho_1^{p_{s-n}-2p}(y) \|u\|_{L^p(I_t(y))}^p \, dy \right),
\]
(4.16)

where \(c_5 \in \mathbb{R}_+\) depends on the same parameters of \(c_4\).

Since
\[
L^p_p(\Omega) \hookrightarrow L^p_{s-1}(\Omega), \quad L^p_{s}(\Omega) \hookrightarrow L^p_{s-2}(\Omega),
\]
(4.17)

from (4.16) and from Lemma 2.1 we have that
\[
\|u\|_{W^{2,p}_t(\Omega)} \leq c_6 \left( \|L_0 u + a'u\|_{L^p_t(\Omega)} + \|u_x\|_{L^p_{s-1}(\Omega)} + \|u\|_{L^p_{s-2}(\Omega)} \right),
\]
(4.18)

with \(c_6 \in \mathbb{R}_+\) dependent on the same parameters of \(c_5\) and also on \(s\).

Moreover, from Lemma 3.2 it follows that for any \(\varepsilon \in \mathbb{R}_+\), there exist \(c'(\varepsilon)\), \(c''(\varepsilon) \in \mathbb{R}_+\), and two bounded open sets \(\Omega'_\varepsilon, \Omega''_\varepsilon \subset \subset \Omega\), both with the cone property, such that
\[
\|u_x\|_{L^p_{s-1}(\Omega)} + \|u\|_{L^p_{s-2}(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}_t(\Omega)} + c'(\varepsilon) \|u\|_{L^p(\Omega'_\varepsilon)},
\]
(4.19)

\[
\left\| \sum_{i=1}^n a_i u_x + a' u \right\|_{L^p_t(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}_t(\Omega)} + c''(\varepsilon) \|u\|_{L^p(\Omega'_\varepsilon)},
\]

where \(c'(\varepsilon)\), \(\Omega'_\varepsilon\) depend on \(\varepsilon\), \(\Omega\), \(n\), \(p\), \(\rho\), \(s\), and \(c''(\varepsilon)\), \(\Omega''_\varepsilon\) depend on \(\varepsilon\), \(\Omega\), \(n\), \(p\), \(t_1\), \(t_2\), \(\rho\), \(s\), \(\sigma_0[a_i]\), and \(\sigma_0[a']\).

From (4.18) and (4.19) it follows (4.6) and then we have the result. \(\square\)

5. A uniqueness result

In this section, we will prove our uniqueness theorem. We begin to prove a regularity result.

**Lemma 5.1.** Suppose that the assumptions \((h_1)\), \((h_2)\), and \((h_3)\) (with \(t_1 > n\) and \(t_2 > n/2\)) hold, and let \(u\) be a solution of the problem

\[
u \in W^{2,q}_{\text{loc}}(\Omega) \cap W^{1,q}_{\text{loc}}(\Omega) \cap L^p_m(\Omega),
\]

\[
Lu \in L^p_q(\Omega),
\]
(5.1)

where \(q \in [1, p]\) and \(m \in \mathbb{R}\). Then, \(u\) belongs to \(W^{2,p}_t(\Omega)\).
Proof. By [13, Lemma 4.1] we have

\[ u \in W^{2,p}_\text{loc}(\Omega) \cap W^{1,p}_\text{loc}(\Omega). \]  

(5.2)

We choose \( r, r' \in \mathbb{R}_+ \), with \( r < r' < 1 \), and a function \( \phi \in C^\infty(\mathbb{R}^n) \), such that

\[ \phi|_{B_r} = 1, \quad \text{supp} \ \phi \subset B_{r'}, \]

\[ \sup_{\mathbb{R}^n} |\partial^a \phi| \leq c_\alpha (r' - r)^{-|a|}, \quad \forall \alpha \in \mathbb{N}_0^n, \]

(5.3)

where \( c_\alpha \in \mathbb{R}_+ \) depends only on \( \alpha \).

We fix \( y \in \mathbb{R}^n \) and put

\[ \psi = \psi_y : x \in \mathbb{R}^n \to \phi\left( \frac{x - y}{\rho_1(y)} \right). \]

(5.4)

Clearly we have

\[ \psi|_{B(y, r\rho_1(y))} = 1, \quad \text{supp} \ \psi \subset B(y, r'\rho_1(y)), \]

\[ \sup_{\mathbb{R}^n} |\partial^a \psi| \leq c_\alpha \rho_1^{-|a|}(y) (r' - r)^{-|a|}, \quad \forall \alpha \in \mathbb{N}_0^n. \]

(5.5)

Since \( \psi u \in W^{2,p}(\Omega) \cap W^{1,p}_\text{loc}(\Omega) \), from [14, Theorem 3.3] it follows that there exist \( c_1 \in \mathbb{R}_+ \) and a bounded open subset \( \Omega_1 \subset \subset \Omega \), with the cone property, such that

\[ \|\psi u\|_{W^{2,p}(\Omega)} \leq c_1 \left( \|L(\psi u)\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)} \right), \]

(5.6)

with \( c_1 \) and \( \Omega_1 \) depending on \( n, p, \Omega, \nu, \mu, g_0, a_{ij}, t, t_1, t_2, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a\|_{L^\infty(\Omega)}, \eta, |\sigma_{ij}|, c_a, c_\sigma, c_a', |\sigma'|, \) where \( t_0 \in \mathbb{R}_+ \) depends on \( n, p, \Omega, \mu, g_0, a_{ij}, t, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|a\|_{L^\infty(\Omega)}, \sigma_{ij}, |\sigma'|, \)

Since

\[ L(\psi u) = -\sum_{i,j=1}^n a_{ij}(\psi u)_{x_i x_j} + \sum_{i=1}^n a_i(\psi u)_{x_i} + a \psi u \]

(5.7)

\[ = \psi Lu - 2 \sum_{i,j=1}^n a_{ij}(\psi_{x_i} u)_{x_j} + \sum_{i,j=1}^n a_{ij}(\psi_{x_i} u)_{x_j} + \sum_{i=1}^n a_i \psi_{x_i} u, \]

from (5.6) and (5.7), we have

\[ \|\psi u\|_{W^{2,p}(\Omega)} \leq c_2 \left( \|\psi Lu\|_{L^p(\Omega)} + \sum_{i,j=1}^n \|\psi_{x_i} u\|_{L^p(\Omega)} + \sum_{i,j=1}^n \|\psi_{x_i} u\|_{L^p(\Omega)} + \sum_{i=1}^n \|a \psi_{x_i} u\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega_1)} \right), \]

(5.8)

with \( c_2 \) dependent on the same parameters of \( c_1 \).
From Lemma 3.1 with \( s = 0 \), we have that
\[
\|a_i q_{x_i} u\|_{L^p(\Omega)} \leq c_3 \|a_i\|_{M^{1,p}(\Omega)} \left( \|q_{x_i} u\|_{L^p(\Omega)} + \|(q_{x_i} u)_{x_i}\|_{L^p(\Omega)} \right),
\]
with \( c_3 \) dependent on \( \Omega, n, p, \) and \( t_1 \).

Using \( \|q_{x_i} u\|_{L^p(\Omega)} + \|(q_{x_i} u)_{x_i}\|_{L^p(\Omega)} \leq c_4 \left( \|q_{x_i} u\|_{L^p(\Omega)} \right)^{1/2} + \|q_{x_i} u\|_{L^p(\Omega)} \),
\[
\|q_{x_i} u\|_{L^p(\Omega)} \leq c_4 \left( \|q_{x_i} u\|_{L^p(\Omega)} \right)^{1/2} + \|q_{x_i} u\|_{L^p(\Omega)}.
\]

where the constant \( c_4 \) depends on \( \Omega, n, p \).

Thus, by (5.8)–(5.10), with easy computations, we deduce the bound:
\[
\|u\|_{W^{2,p}(L^p(\Omega))} \leq \|\psi u\|_{W^{2,p}(\Omega)} \leq c_5 (r' - r)^{-2}
\times \left( \|Lu\|_{L^p(I,y)} + \|u\|^p_{W^{2,p}(L^p(\Omega))} \right)^{1/2} \left( \rho_1^{-1}(y) \|u\|_{L^p(I,y)} \right)^{1/2} + \rho_1^{-1}(y) \|u\|_{L^p(I,y)},
\]

where \( c_5 \in \mathbb{R}_+ \) depends on \( n, p, \rho, \Omega, \eta, \mu, g_0, a_0^n, t, t_1, t_2, \|a_{ij}\|_{L^p(\Omega)}, \|e_{ij}\|_{L^p(\Omega)}, \|\theta\|_{L^p(\Omega)}, \|a''\|_{L^p(\Omega)}, \eta_{ij} a_{ij} \), \( \sigma_i [e_{ij}, (e_{ij})_x], \|a_{ij}\|_{M^{1,p}(\Omega)}, \sigma_i [a_i], \sigma_i [a'] \).

By a well-known monotonicity of Miranda (see [23, Lemma 3.1]), it follows from (5.11) that
\[
\|u\|_{W^{2,p}(L^p(I,y))} \leq c_6 \left( \|Lu\|_{L^p(I,y)} + \rho_1^{-1}(y) \|u\|_{L^p(I,y)} \right)^{1/2} \|u\|_{W^{2,p}(L^p(I,y))}.
\]

and then, using Young’s inequality, we deduce from (5.12) that
\[
\|u\|_{W^{2,p}(L^p(I,y))} \leq c_7 \left( \|Lu\|_{L^p(I,y)} + \rho_1^{-1}(y) \|u\|_{L^p(I,y)} \right),
\]

with \( c_7 \in \mathbb{R}_+ \) dependent on the same parameters of \( c_5 \).

From (5.13) it follows
\[
\int_{\mathbb{R}^n} \rho_1^{p-1}(y) \|u\|_{W^{2,p}(L^p(I,y))}^p \, dy 
\leq c_8 \left( \int_{\mathbb{R}^n} \rho_1^{p-1}(y) \|Lu\|_{L^p(I,y)}^p \, dy \right) \left( \int_{\mathbb{R}^n} \rho_1^{p-1}(y) \|u\|_{L^p(I,y)}^p \, dy \right),
\]

where \( c_8 \in \mathbb{R}_+ \) depends on the same parameters of \( c_7 \).

If \( m \geq s - 1 \), since
\[
L^{p_m}_m(\Omega) \hookrightarrow L^{p_{s-1}}(\Omega),
\]
from (5.14) and from Lemma 2.1 we have that
\[
\|u\|_{W^{2,p}_s(\Omega)} \leq c_9 \left( \|Lu\|_{L^{p_m}_m(\Omega)} + \|u\|_{L^{p_{s-1}}(\Omega)} \right),
\]

with \( c_9 \in \mathbb{R}_+ \) dependent on the same parameters of \( c_8 \) and on \( s \). Therefore, \( u \) belongs to \( W^{2,p}_s(\Omega) \).
If $m < s - 1$, we denote by $k$ the positive integer, such that
\begin{equation}
    s - m - 1 \leq k < s - m. \tag{5.17}
\end{equation}
Then, for $i = 1, \ldots, k$, we have that
\begin{equation}
    L^p_s(\Omega) \hookrightarrow L^p_{m+i}(\Omega). \tag{5.18}
\end{equation}
Therefore, using (5.14) and (5.16) with $m + i$, $i = 1, \ldots, k$, instead of $s$, we deduce that $u \in W_{m+1}^{2,p}(\Omega), \ldots, u \in W_{m+k}^{2,p}(\Omega)$. On the other hand, we have that
\begin{equation}
    W_{m+k}^{2,p}(\Omega) \hookrightarrow L^p_{s-1}(\Omega) \tag{5.19}
\end{equation}
and then, since $u \in L^p_{s-1}(\Omega)$, (5.14) holds. Thus, $u$ satisfies (5.16) and then $u \in W_{s}^{2,p}(\Omega)$.

**Theorem 5.2.** If conditions $(h_1)$, $(h_2)$, and $(h_3)$ (with $t_2 > n$ and $t_2 > n/2$) hold, and $a \geq a_0 > 0$ a.e. in $\Omega$, then the problem
\begin{equation}
    u \in W_{s}^{2,p}(\Omega) \cap W^{1,p}_{s}(\Omega), \quad Lu = 0, \tag{5.20}
\end{equation}
admits only the zero solution.

**Proof.** Fix $u \in W_{s}^{2,p}(\Omega) \cap W^{1,p}_{s}(\Omega)$, such that $Lu = 0$. From Lemma 5.1 it follows that $u \in W^{2,p}(\Omega)$. On the other hand, since $u \in W^{1,p}(\Omega) \cap W^{1,p}_{loc}(\Omega)$, from Lemma 2.2 we have that $u \in W_{s}^{1,p}, \ldots, k$. Thus, from [13, Theorem 5.2] we deduce that $u = 0$. $\blacksquare$

**References**


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